Sparsest Cut

1 Introduction

SPARSEST CUT is a fundamental problem in graph algorithms with connections to various cut related problems.

Problem 1 (NON-UNIFORM SPARSEST CUT) *The input is a graph $G = (V, E)$ with edge capacities $c : E \to \mathbb{R}_+$ and a set of vertex pairs $\{s_1, t_1\}, \dots, \{s_k, t_k\}$ along with demand values $D_1, \dots, D_k \in \mathbb{R}_+$. The goal is to find a cut $\delta(S)$ of G such that $\frac{c(\delta(S))}{\sum_{i:|S\cap\{s_i,t_i\}|=1} D_i}$ is minimized.*

In other words, NON-UNIFORM SPARSEST CUT finds the cut that minimizes its capacity divided by the sum of demands of the vertex pairs it separates. There are two important varients of NON-UNIFORM SPARSEST CUT. Note that we always consider unordered pair $\{s_i, t_i\}$, i.e., we do not distinguish $\{s_i, t_i\}$ and $\{t_i, s_i\}$.

SPARSEST CUT is the uniform version of NON-UNIFORM SPARSEST CUT. The demand is 1 for every possible vertex pair $\{s_i, t_i\}$. In this case, we can remove from the input the pairs and demands. The goal becomes to minimize $\frac{c(\delta(S))}{|S||V\setminus S|}$.

EXPANSION further simplifies the objective of SPARSEST CUT to $\min_{|S|\le n/2} \frac{c(\delta(S))}{|S|}$.

These problems are interesting since they are related to central concepts in graph theory and help to design algorithms for hard problems on graph. One connections is expander graphs. The importance of expander graphs is thoroughly surveyed in [HLW06]. The optimum of EXPANSION is also known as Cheeger constant or conductance of a graph. SPARSEST CUT provides a 2-approximation of Cheeger constant, which is especially important in the context of expander graphs as it is a way to measure the edge expansion of a graph. NON-UNIFORM SPARSEST CUT is related to other cut problems such as Multicut and Balanced Separator. From a more mathematical perspective, the techniques developed for approximating SPARSEST CUT are deeply related to metric embedding, which is another fundamental problem in geometry. Besides theoretical interests, SPARSEST CUT is useful in practical scenarios such as in image segmentation and in some machine leaning algorithms.

1.1 related works

NON-UNIFORM SPARSEST CUT is APX-hard [CK09] and, assuming the Unique Game Conjecture, has no polynomial time constant factor aproximation algorithm[CKK+05]. SPARSEST CUT admits no PTAS [AMS07], assuming a widely believed conjecture. The currently best approximation algorithm for SPARSEST CUT has ratio $O(\sqrt{\log n})$ and running time $\tilde{O}(n^2)$ [AHK10]. Prior to this currently optimal result, there is a long line of research optimizing both the approximation ratio and the complexity, see [ARV04, LR99]. There are also works concerning approximating SPARSEST CUT on special graph classes such as planar graphs [LS10], graphs with low treewidth [CKR10, GTW13, CKM+24].

For an overview of the LP methods for SPARSEST CUT, see https://courses.grainger.illinois.edu/cs598csc/fa2024/Notes/lec-sparsest-cut.pdf.

The seminal work of [LR99] starts this line of research. They studied multicommodity flow problem and proved a $O(\log n)$ flow-cut gap (in fact the tight $\Theta(\log n)$ gap was proven by [AR95] and [LLR95]).

They also developed $O(\log n)$ approximation algorithm for multicommodity flow problems, which can imply $O(\log n)$ approximation for SPARSEST CUT and $O(\log^2 n)$ approximation for NON-UNIFORM SPARSEST CUT. The technique is called region growing. They also discovered a lowerbound of $\Omega(\log n)$ via expanders. Note that any algorithm achieving the $O(\log n)$ flow cut gap implies an $O(\log^2 n)$ approximation for NON-UNIFORM SPARSEST CUT, but better ratio is still possible through other methods. This paper showed that $O(\log^2 n)$ is the best approximation we can achieve using flow-cut gap.

For NON-UNIFORM SPARSEST CUT [LR99] only guarantees a $O(\log^2 n)$ approximation. This is further improved by [LLR95] and [AR98]. [AR98] applied metric embedding to NON-UNIFORM SPARSEST CUT and obtained a $O(\log n)$ approximation. The connections between metric embedding and NON-UNIFORM SPARSEST CUT is influential. NON-UNIFORM SPARSEST CUT can be formulated as an integer program. [AR98], [AR95] and [LLR95] considered the metric relaxation of the IP. They observed that NON-UNIFORM SPARSEST CUT is polynomial time solvable for trees and more generally for all ℓ_1 metrics. The $O(\log n)$ approximation follows from the $O(\log n)$ distortion in the metric embedding theorem.

[ARV04] and [AHK10] further improved the approximation ratio for SPARSEST CUT to $O(\sqrt{\log n})$ via semidefinite relaxation. This is currently the best approximation ratio for SPARSEST CUT.

There is also plenty of research concerning SPARSEST CUT on some graph classes, for example [BBPP12]. One of the most popular class is graphs with constant treewidth. [CKM+24] gave a $O(k^2)$ approximation algorithm with complexity $2^{O(k)}\operatorname{poly}(n)$. [CAMV24] obtained a 2-approximation algorithm for sparsest cut in treewidth k graph with running time $2^{2^{O(k)}}\operatorname{poly}(n)$.

SPARSEST CUT is easy on trees and the flow-cut gap is 1 for trees. One explaination[1] is that shortest path distance in trees is an ℓ_1 metric. There are works concerning planar graphs and more generally graphs with constant genus. [LR99] provided a $\Omega(\log n)$ lowerbound for flow-cut gap for SPARSEST CUT. However, it is conjectured that the gap is $O(1)$, while currently the best upperbound is still $O(\sqrt{\log n})$ [Rao99]. For graphs with constant genus, [LS10] gives a $O(\sqrt{\log g})$ approximation for SPARSEST CUT, where g is the genus of the input graph. For flow-cut gap in planar graphs the techniques are mainly related to metric embedding theory[2].

2 Approximations

Techniques for approximating uniform SPARSEST CUT and NON-UNIFORM SPARSEST CUT.

2.1 LP $\Theta(\log n)$

$$\begin{aligned} \min \quad & \frac{\sum_e c_e x_e}{\sum_i D_i y_i} \\ s.t. \quad & \sum_{e \in p} x_e \geq y_i \qquad \forall p \in \mathcal{P}_{s_i,t_i}, \forall i \\ & x_e, y_i \in \{0,1\} \end{aligned} \tag{1}$$

$$\begin{aligned} \min \quad & \sum_e c_e x_e \\ s.t. \quad & \sum_i D_i y_i = 1 \\ & \sum_{e \in p} x_e \geq y_i \quad \forall p \in \mathcal{P}_{s_i,t_i}, \forall i \\ & x_e, y_i > 0 \end{aligned} \tag{2}$$

[1] https://courses.grainger.illinois.edu/cs598csc/fa2024/Notes/lec-sparsest-cut.pdf

[2] https://home.ttic.edu/~harry/teaching/teaching.html

$$\begin{aligned} \max \quad & \lambda \\ s.t. \quad & \sum_{p\in\mathcal{P}_{s_i,t_i}} y_p \geq \lambda D_i \quad \forall i \\ & \sum_i \sum_{p\in\mathcal{P}_{s_i,t_i}, p\ni e} y_p \geq c_e \quad \forall e \\ & y_p \geq 0 \end{aligned} \tag{3}$$

$$\begin{aligned} \min \quad & \sum_{uv\in E} c_{uv} d(u,v) \\ s.t. \quad & \sum_i D_i d(s_i,t_i) = 1 \\ & d \text{ is a metric on } V \end{aligned} \tag{4}$$

1. IP1 $\geq$ LP2. Given any feasible solution to IP1, we can scale all x_e and y_i simultaneously with factor $1/\sum_i D_i y_i$. The scaled solution is feasible for LP2 and gets the same objective value.

2. LP2 $=$ LP3. by duality.

3. LP4 $=$ LP2. It is easy to see LP4 $\geq$ LP2 since any feasible metric to LP4 induces a feasible solution to LP2. In fact, the optimal solution to LP2 also induces a feasible metric. Consider a solution x_e, y_i to LP2. Let d_x be the shortest path metric on V using edge length x_e. It suffices to show that $y_i = d_x(s_i,t_i)$. This can be seen from a reformulation of LP2. The constraint $\sum_i D_i y_i = 1$ can be removed and the objective becomes $\sum_e c_e x_e / \sum_i D_i y_i$. This reformulation does not change the optimal solution. Now suppose in the optimal solution to LP2 there is some y_i which is strictly smaller than $d_x(s_i,t_i)$. Then the denominator $\sum_i D_i y_i$ in the objective of our reformulation can be larger, contradicting to the optimality of solution x_e, y_i.

Theorem 2.1 (Japanese Theorem) *D is a demand matrix. D is routable in G iff $\forall l : E \to \mathbb{R}_+$, $\sum_e c_e l(e) \geq \sum_{uv} D(u,v) d_l(u,v)$, where $d_l(s,t)$ is the short path distance induced by $l(e)$.*

Note that D is routable iff the optimum of the LPs is at least 1. Then the theorem follows directly from LP4.

$\Theta(\log n)$ flow-cut gap The flow-cut gap is OPT(IP1)/OPT(LP2) [LR99].

Suppose that G satisfies the cut condition, that is, $c(\delta(S))$ is at least the demand separated by $\delta(S)$ for all $S \subseteq V$. This implies OPT(IP1) ≥ 1 and in this case the largest integrality gap is 1/OPT(LP2). For 1 and 2-commodity flow problem the gap is 1 [FF56, Hu63]. However, for $k \geq 3$ the gap becomes larger[3]. It is mentioned in [LR99] that [Sch90] proved if the demand graph does not contain either three disjoint edges or a triangle and a disjoint edge, then the gap is 1.

For the $\Omega(\log n)$ lowerbound consider an uniform Sparsest Cut instance on some 3-regular graph G with unit capacity. In [LR99] they further required that for any $S \subseteq V$ and small constant c, $|\delta(S)| \geq c\min(|S|, |\bar{S}|)$. Then the value of the sparsest cut is at least $\frac{c}{n-1}$. Observe that for any fixed vertex v, there are at most $n/2$ vertices within distance $\log n - 3$ of v. Thus at least half of the $\binom{n}{2}$ demand pairs are connected with shortest path of length at least $\log n - 2$. To sustain a flow f we need at least $\frac{1}{2}\binom{n}{2}(\log n - 2) f \leq 3n/2$. Any feasible flow satisfies $f \leq \frac{3n}{\binom{n}{2}(\log n - 2)}$ and the gap is therefore $\Omega(\log n)$.

For the upperbound it suffices to show there exists a cut of ratio $O(f \log n)$. [LR99] gave an algorithmic proof based on LP4. This can also be proven using metric embedding results. We can solve LP4 in polynomial time and get a metric on V. Then there is an embedding of V into $\mathbb{R}^d$ with ℓ_1 metric such that the distortion is $O(\log n)$. Since ℓ_1 metric is in the cut cone, our metric on $\mathbb{R}^d$ is a conic combination of cut metrics, which implies[4] that there is a cut in the conic combination with value at most $O(\log n)$ OPT(LP4). To find such a cut it suffices to compute a conic combination of cut metrics which is exactly our ℓ_1 metric in $\mathbb{R}^d$. One way to do this is test $(n-1)d$ cuts by observing the followings,

[3] https://en.wikipedia.org/wiki/Approximate_max-flow_min-cut_theorem

[4] This requires some work. See https://courses.grainger.illinois.edu/cs598csc/fa2024/Notes/lec-sparsest-cut.pdf

1. Every coordinate of $\mathbb{R}^d$ corresponds to a line metric;
2. ℓ_1 metric in $\mathbb{R}^d$ is the sum of those line metrics;
3. Every line metric on n points can be represented as some conic combination of $n-1$ cut metrics.

Remark I believe the later method is more general and works for NON-UNIFORM SPARSEST CUT, while the former method is limited to uniform SPARSEST CUT. However, the proof in [LR99] may have connections with the proof of Bourgain's thm? why does the method in [LR99] fail to work on NON-UNIFORM SPARSEST CUT?

2.2 SDP $O(\sqrt{\log n})$

SDP approximation follows from metric embedding results.

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