Approximation Algorithms for Sparsest Cut

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Major: Theoretical Computer Science

Research problem What is the best possible approximation rate of linear programming based approximation algorithms for Sparsest Cut? What about algorithms for planar graphs?

1 Introduction

Sparsest Cut is a fundamental problem in graph algorithms with connections to various cut related problems.

Problem 1 (Non-Uniform Sparsest Cut) *The input is a graph $G = (V, E)$ with edge capacities $c : E \to \mathbb{R}_+$ and a set of vertex pairs $\{s_1, t_1\}, \dots, \{s_k, t_k\}$ along with demand values $D_1, \dots, D_k \in \mathbb{R}_+$. The goal is to find a cut $\delta(S)$ of G such that $\frac{c(\delta(S))}{\sum_{i:|S\cap\{s_i,t_i\}|=1} D_i}$ is minimized.*

In other words, Non-Uniform Sparsest Cut finds the cut that minimizes its capacity divided by the sum of demands of the vertex pairs it separates. There are two important varients of Non-Uniform Sparsest Cut. Note that we always consider unordered pair $\{s_i, t_i\}$, i.e., we do not distinguish $\{s_i, t_i\}$ and $\{t_i, s_i\}$.

Sparsest Cut is the uniform version of Non-Uniform Sparsest Cut. The demand is 1 for every possible vertex pair $\{s_i, t_i\}$. In this case, we can remove from the input the pairs and demands. The goal becomes to minimize $\frac{c(\delta(S))}{|S||V\setminus S|}$.

Expansion further simplifies the objective of Sparsest Cut to $\min_{|S|\le n/2} \frac{c(\delta(S))}{|S|}$.

These problems are interesting since they are related to central concepts in graph theory and help to design algorithms for hard problems on graph. One connections is expander graphs. The importance of expander graphs is thoroughly surveyed in [Hoory et al., 2006]. The optimum of Expansion is also known as Cheeger constant or conductance of a graph. Sparsest Cut provides a 2-approximation of Cheeger constant, which is especially important in the context of expander graphs as it is a way to measure the edge expansion of a graph. Non-Uniform Sparsest Cut is related to other cut problems such as Multicut and Balanced Separator.

1.1 related works

Sparsest Cut is APX-hard [Chuzhoy and Khanna, 2009] and, assuming the Unique Game Conjecture, has no polynomial time constant factor aproximation algorithm[Chawla et al., 2005]. The currently best approximation algorithm has ratio $O(\sqrt{\log n})$ and running time $\tilde{O}(n^2)$ [Arora et al., 2010]. Prior to this currently optimal result, there is a long line of research optimizing both the approximation ratio and the complexity, see [Arora et al., 2004, Leighton and Rao, 1999]. There are also works concerning approximating Sparsest Cut on special graph classes such as planar graphs [Lee and Sidiropoulos, 2010], graphs with low tree width [Chalermsook et al., 2024, Gupta et al., 2013].

One major open problem for Sparsest Cut is the best approximation ratio for planar graphs. It is conjectured that the ratio for planar graphs is $O(1)$ but currently the best lowerbound is $O(\sqrt{\log n})$.

2 Literature Review

3 The Research Design

4 Time Table

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