

# Sparsest Cut

## 1 Introduction

SPARSEST CUT is a fundamental problem in graph algorithms with connections to various cut related problems.

**Problem 1 (NON-UNIFORM SPARSEST CUT)** *The input is a graph  $G = (V, E)$  with edge capacities  $c : E \rightarrow \mathbb{R}_+$  and a set of vertex pairs  $\{s_1, t_1\}, \dots, \{s_k, t_k\}$  along with demand values  $D_1, \dots, D_k \in \mathbb{R}_+$ . The goal is to find a cut  $\delta(S)$  of  $G$  such that  $\frac{c(\delta(S))}{\sum_{i: |S \cap \{s_i, t_i\}|=1} D_i}$  is minimized.*

In other words, NON-UNIFORM SPARSEST CUT finds the cut that minimizes its capacity divided by the sum of demands of the vertex pairs it separates. There are two important variants of NON-UNIFORM SPARSEST CUT. Note that we always consider unordered pair  $\{s_i, t_i\}$ , i.e., we do not distinguish  $\{s_i, t_i\}$  and  $\{t_i, s_i\}$ .

SPARSEST CUT is the uniform version of NON-UNIFORM SPARSEST CUT. The demand is 1 for every possible vertex pair  $\{s_i, t_i\}$ . In this case, we can remove from the input the pairs and demands. The goal becomes to minimize  $\frac{c(\delta(S))}{|S||V \setminus S|}$ .

EXPANSION further simplifies the objective of SPARSEST CUT to  $\min_{|S| \leq n/2} \frac{c(\delta(S))}{|S|}$ .

### 1.1 importance and connections

These problems are interesting since they are related to central concepts in graph theory and help to design algorithms for hard problems on graph. One connections is expander graphs. The importance of expander graphs is thoroughly surveyed in [Hoory et al., 2006]. The optimum of EXPANSION is also known as Cheeger constant or conductance of a graph. SPARSEST CUT provides a 2-approximation of Cheeger constant, which is especially important in the context of expander graphs as it is a way to measure the edge expansion of a graph. NON-UNIFORM SPARSEST CUT is related to other cut problems such as Multicut and Balanced Separator. From a more mathematical perspective, the techniques developed for approximating SPARSEST CUT are deeply related to metric embedding, which is another fundamental problem in geometry. Besides theoretical interests, SPARSEST CUT is useful in practical scenarios such as in image segmentation and in some machine learning algorithms.

### 1.2 related works

NON-UNIFORM SPARSEST CUT is APX-hard [Chuzhoy and Khanna, 2009] and, assuming the Unique Game Conjecture, has no polynomial time constant factor approximation algorithm [Chawla et al., 2005]. SPARSEST CUT admits no PTAS [Ambuhl et al., 2007], assuming a widely believed conjecture. The currently best approximation algorithm has ratio  $O(\sqrt{\log n})$  and running time  $\tilde{O}(n^2)$  [Arora et al., 2010]. Prior to this currently optimal result, there is a long line of research optimizing both the approximation ratio and the complexity, see [Arora et al., 2004, Leighton and Rao, 1999]. There are also works concerning approximating SPARSEST CUT on special graph classes such as planar graphs [Lee and Sidiropoulos, 2010], graphs with low treewidth [Chlamtac et al., 2010, Gupta et al., 2013, Chalermsook et al., 2024].

For an overview of the LP methods for SPARSEST CUT, see [Chekuri, 2024].

### 1.3 open problems

One major open problem for SPARSEST CUT is the best approximation ratio for planar graphs. It is conjectured that the ratio for planar graphs is  $O(1)$  but currently the best lowerbound is  $O(\sqrt{\log n})$ . For graphs treewidth  $k$ , an open problem is that whether there is a 2 approximation algorithm that runs in  $2^{O(k)} \text{poly}(n)$ .

## 2 Literature Review

The seminal work of [Leighton and Rao \[1999\]](#) starts this line of research. They studied multicommodity flow problem and proved a  $O(\log n)$  flow-cut gap. They also developed  $O(\log n)$  approximation algorithm for multicommodity flow problems, which can imply  $O(\log n)$  approximation for SPARSEST CUT and  $O(\log^2 n)$  approximation for NON-UNIFORM SPARSEST CUT. The technique is called region growing. They also discovered a lowerbound of  $\Omega(\log n)$  via expanders. Note that any algorithm achieving the  $O(\log n)$  flow cut gap implies an  $O(\log^2 n)$  approximation for NON-UNIFORM SPARSEST CUT, but better ratio is still possible through other methods. This paper showed that  $O(\log^2 n)$  is the best approximation we can achieve using flow-cut gap.

For NON-UNIFORM SPARSEST CUT [[Leighton and Rao, 1999](#)] only guarantees a  $O(\log^2 n)$  approximation. This is further improved by [[Linial et al., 1995](#)] and [[Aumann and Rabani, 1998](#)]. [Aumann and Rabani \[1998\]](#) applied metric embedding to NON-UNIFORM SPARSEST CUT and obtained a  $O(\log n)$  approximation. The connections between metric embedding and NON-UNIFORM SPARSEST CUT is influential. NON-UNIFORM SPARSEST CUT can be formulated as an integer program. [Aumann and Rabani](#) considered the metric relaxation of the IP. They observed that NON-UNIFORM SPARSEST CUT is polynomial time solvable for trees and more generally for all  $\ell_1$  metrics. The  $O(\log n)$  approximation follows from the  $O(\log n)$  distortion in the metric embedding theorem.

[[Arora et al., 2004](#)] and [[Arora et al., 2010](#)] further improved the approximation ratio for SPARSEST CUT to  $O(\sqrt{\log n})$  via semidefinite relaxation. This is currently the best approximation ratio for SPARSEST CUT.

There is also plenty of research concerning SPARSEST CUT on some graph classes, for example [[Bonsma et al., 2012](#)]. One of the most popular class is graphs with constant treewidth. [[Chalermsook et al., 2024](#)] gave a  $O(k^2)$  approximation algorithm with complexity  $2^{O(k)} \text{poly}(n)$ . [[Cohen-Addad et al., 2024](#)] obtained a 2-approximation algorithm for sparsest cut in treewidth  $k$  graph with running time  $2^{2^{O(k)}} \text{poly}(n)$ .

SPARSEST CUT is easy on trees and the flow-cut gap is 1 for trees. One explanation mentioned in [[Chekuri, 2024](#)] is that shortest path distance in trees is an  $\ell_1$  metric. There are works concerning planar graphs and more generally graphs with constant genus. [[Leighton and Rao, 1999](#)] provided a  $\Omega(\log n)$  lowerbound for flow-cut gap for SPARSEST CUT. However, it is conjectured that the gap is  $O(1)$ , while currently the best upperbound is still  $O(\sqrt{\log n})$  [[Rao, 1999](#)]. For graphs with constant genus, [[Lee and Sidiropoulos, 2010](#)] gives a  $O(\sqrt{\log g})$  approximation for SPARSEST CUT, where  $g$  is the genus of the input graph. For flow-cut gap in planar graphs the techniques are mainly related to metric embedding theory.

### 3 LP

$$\begin{aligned}
& \min \quad \frac{\sum_e c_e x_e}{\sum_i D_i y_i} \\
& \text{s.t.} \quad \sum_{e \in p} x_e \geq y_i \quad \forall \mathcal{P}_{s_i, t_i}, \forall i \quad (1) \\
& \quad \quad x_e, y_i \in \{0, 1\}
\end{aligned}$$

$$\begin{aligned}
& \min \quad \sum_e c_e x_e \\
& \text{s.t.} \quad \sum_i D_i y_i = 1 \\
& \quad \quad \sum_{e \in p} x_e \geq y_i \quad \forall \mathcal{P}_{s_i, t_i}, \forall i \quad (2) \\
& \quad \quad x_e, y_i > 0
\end{aligned}$$

$$\begin{aligned}
& \max \quad \lambda \\
& \text{s.t.} \quad \sum_{p \in \mathcal{P}_{s_i, t_i}} y_p \geq \lambda D_i \quad \forall i \\
& \quad \quad \sum_i \sum_{p \in \mathcal{P}_{s_i, t_i}, p \ni e} y_p \geq c_e \quad \forall e \quad (3) \\
& \quad \quad y_p \geq 0
\end{aligned}$$

$$\begin{aligned}
& \min \quad \sum_{uv \in E} c_{uv} d(u, v) \\
& \text{s.t.} \quad \sum_i D_i d(s_i, t_i) = 1 \quad (4) \\
& \quad \quad d \text{ is a metric on } V
\end{aligned}$$

1.  $\text{IP1} \geq \text{LP2}$ . Given any feasible solution to  $\text{IP1}$ , we can scale all  $x_e$  and  $y_i$  simultaneously with factor  $1/\sum_i D_i y_i$ . The scaled solution is feasible for  $\text{LP2}$  and gets the same objective value.
2.  $\text{LP2} = \text{LP3}$ . by duality.
3.  $\text{LP4} = \text{LP2}$ . It is easy to see  $\text{LP4} \geq \text{LP2}$  since any feasible metric to  $\text{LP4}$  induces a feasible solution to  $\text{LP2}$ . In fact, any solution to  $\text{LP2}$  also induces a feasible metric.

### References

- Christoph Ambuhl, Monaldo Mastrolilli, and Ola Svensson. Inapproximability results for sparsest cut, optimal linear arrangement, and precedence constrained scheduling. In *48th Annual IEEE Symposium on Foundations of Computer Science (FOCS'07)*, pages 329–337, 2007. doi: 10.1109/FOCS.2007.40.
- Sanjeev Arora, Satish Rao, and Umesh Vazirani. Expander flows, geometric embeddings and graph partitioning. In *Proceedings of the thirty-sixth annual ACM symposium on Theory of computing*, STOC '04, pages 222–231, New York, NY, USA, June 2004. Association for Computing Machinery. ISBN 978-1-58113-852-8. doi: 10.1145/1007352.1007355. URL <https://doi.org/10.1145/1007352.1007355>.
- Sanjeev Arora, Elad Hazan, and Satyen Kale.  $O(\sqrt{\log n})$  Approximation to SPARSEST CUT in  $\tilde{O}(n^2)$  Time. *SIAM Journal on Computing*, 39(5):1748–1771, January 2010. ISSN 0097-5397, 1095-7111. doi: 10.1137/080731049. URL <http://epubs.siam.org/doi/10.1137/080731049>.
- Yonatan Aumann and Yuval Rabani. An  $O(\log k)$  approximate min-cut max-flow theorem and approximation algorithm. *SIAM Journal on Computing*, 27(1):291–301, 1998. doi: 10.1137/S0097539794285983. URL <https://doi.org/10.1137/S0097539794285983>.
- Paul Bonsma, Hajo Broersma, Viresh Patel, and Artem Pyatkin. The complexity of finding uniform sparsest cuts in various graph classes. *Journal of Discrete Algorithms*, 14:136–149, July 2012. ISSN 15708667. doi: 10.1016/j.jda.2011.12.008. URL <https://linkinghub.elsevier.com/retrieve/pii/S1570866711001110>.

- Parinya Chalermsook, Matthias Kaul, Matthias Mnich, Joachim Spoerhase, Sumedha Uniya, and Daniel Vaz. Approximating sparsest cut in low-treewidth graphs via combinatorial diameter. *ACM Transactions on Algorithms*, 20(1):1–20, January 2024. ISSN 1549-6333. doi: 10.1145/3632623. URL <http://dx.doi.org/10.1145/3632623>.
- S. Chawla, R. Krauthgamer, R. Kumar, Y. Rabani, and D. Sivakumar. On the hardness of approximating MULTICUT and SPARSEST-CUT. In *20th Annual IEEE Conference on Computational Complexity (CCC'05)*, pages 144–153, June 2005. doi: 10.1109/CCC.2005.20. URL <https://ieeexplore.ieee.org/document/1443081>. ISSN: 1093-0159.
- Chandra Chekuri. Introduction to sparsest cut. Lecture notes, UIUC CS 598CSC: Topics in Graph Algorithms, 2024. URL <https://courses.grainger.illinois.edu/cs598csc/fa2024/Notes/lec-sparsest-cut.pdf>. Accessed on May 9, 2025.
- Eden Chlamtac, Robert Krauthgamer, and Prasad Raghavendra. Approximating Sparsest Cut in Graphs of Bounded Treewidth. In Maria Serna, Ronen Shaltiel, Klaus Jansen, and José Rolim, editors, *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*, pages 124–137, Berlin, Heidelberg, 2010. Springer. ISBN 978-3-642-15369-3. doi: 10.1007/978-3-642-15369-3\_10.
- Julia Chuzhoy and Sanjeev Khanna. Polynomial flow-cut gaps and hardness of directed cut problems. *J. ACM*, 56(2), April 2009. ISSN 0004-5411. doi: 10.1145/1502793.1502795. URL <https://doi.org/10.1145/1502793.1502795>.
- Vincent Cohen-Addad, Tobias Mömke, and Victor Verdugo. A 2-approximation for the bounded treewidth sparsest cut problem in FPTtime. *Mathematical Programming*, 206(1):479–495, July 2024. ISSN 1436-4646. doi: 10.1007/s10107-023-02044-1. URL <https://doi.org/10.1007/s10107-023-02044-1>.
- Anupam Gupta, Kunal Talwar, and David Witmer. Sparsest cut on bounded treewidth graphs: Algorithms and hardness results, 2013. URL <https://arxiv.org/abs/1305.1347>.
- Shlomo Hoory, Nathan Linial, and Avi Wigderson. Expander graphs and their applications. *Bulletin of the American Mathematical Society*, 43(04):439–562, August 2006. ISSN 0273-0979. doi: 10.1090/S0273-0979-06-01126-8. URL <http://www.ams.org/journal-getitem?pii=S0273-0979-06-01126-8>.
- James R. Lee and Anastasios Sidiropoulos. Genus and the geometry of the cut graph: [extended abstract]. In *Proceedings of the Twenty-First Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 193–201. Society for Industrial and Applied Mathematics, January 2010. ISBN 978-0-89871-701-3 978-1-61197-307-5. doi: 10.1137/1.9781611973075.18. URL <https://epubs.siam.org/doi/10.1137/1.9781611973075.18>.
- Tom Leighton and Satish Rao. Multicommodity max-flow min-cut theorems and their use in designing approximation algorithms. *J. ACM*, 46(6):787–832, November 1999. ISSN 0004-5411. doi: 10.1145/331524.331526. URL <https://dl.acm.org/doi/10.1145/331524.331526>.
- Nathan Linial, Eran London, and Yuri Rabinovich. The geometry of graphs and some of its algorithmic applications. *Combinatorica*, 15(2):215–245, June 1995. ISSN 0209-9683, 1439-6912. doi: 10.1007/BF01200757. URL <http://link.springer.com/10.1007/BF01200757>.
- Satish Rao. Small distortion and volume preserving embeddings for planar and Euclidean metrics. In *Proceedings of the fifteenth annual symposium on Computational geometry*, pages 300–306, Miami Beach Florida USA, June 1999. ACM. ISBN 978-1-58113-068-3. doi: 10.1145/304893.304983. URL <https://dl.acm.org/doi/10.1145/304893.304983>.