Approximation Algorithms for Sparsest Cut

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Major: Theoretical Computer Science

Research problem   What is the best possible approximation rate of linear programming based approximation algorithms for SPARSEST CUT? What about algorithms for planar graphs?

1   Introduction

SPARSEST CUT is a fundamental problem in graph algorithms with connections to various cut related problems.

Problem 1 (NON-UNIFORM SPARSEST CUT) *The input is a graph $G = (V, E)$ with edge capacities $c : E \to$ $\mathbb{R}_+$ and a set of vertex pairs $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$ along with demand values $D_1, \ldots, D_k \in \mathbb{R}_+$. The goal is to find a cut $\delta(S)$ of G such that $\frac{c(\delta(S))}{\sum_{i:|S \cap \{s_i, t_i\}|=1} D_i}$ is minimized.*

In other words, NON-UNIFORM SPARSEST CUT finds the cut that minimizes its capacity divided by the sum of demands of the vertex pairs it separates. There are two important varients of NON-UNIFORM SPARSEST CUT. Note that we always consider unordered pair $\{s_i, t_i\}$, i.e., we do not distinguish $\{s_i, t_i\}$ and $\{t_i, s_i\}$.

SPARSEST CUT is the uniform version of NON-UNIFORM SPARSEST CUT. The demand is 1 for every possible vertex pair $\{s_i, t_i\}$. In this case, we can remove from the input the pairs and demands. The goal becomes to minimize $\frac{c(\delta(S))}{|S||V \setminus S|}$.

EXPANSION further simplifies the objective of SPARSEST CUT to $\min_{|S| \leq n/2} \frac{c(\delta(S))}{|S|}$.

1.1   importance and connections

These problems are interesting since they are related to central concepts in graph theory and help to design algorithms for hard problems on graph. One connections is expander graphs. The importance of expander graphs is thoroughly surveyed in [Hoory et al., 2006]. The optimum of EXPANSION is also known as Cheeger constant or conductance of a graph. SPARSEST CUT provides a 2-approximation of Cheeger constant, which is especially important in the context of expander graphs as it is a way to measure the edge expansion of a graph. NON-UNIFORM SPARSEST CUT is related to other cut problems such as Multicut and Balanced Separator. From a more mathematical perspective, the techniques developed for approximating SPARSEST CUT are deeply related to metric embedding, which is another fundamental problem in geometry. Besides theoretical interests, SPARSEST CUT is useful in practical scenarios such as in image segmentation and in some machine leaning algorithms.

1.2   related works

SPARSEST CUT is APX-hard [Chuzhoy and Khanna, 2009] and, assuming the Unique Game Conjecture, has no polynomial time constant factor aproximation algorithm[Chawla et al., 2005]. The currently best approximation algorithm has ratio $O(\sqrt{\log n})$ and running time $\tilde{O}(n^2)$ [Arora et al., 2010]. Prior to this currently optimal result, there is a long line of research optimizing both the approximation ratio and the complexity, see [Arora et al., 2004, Leighton and Rao, 1999]. There are also works concerning

approximating Sparsest Cut on special graph classes such as planar graphs [Lee and Sidiropoulos, 2010], graphs with low treewidth [Chlamtac et al., 2010, Gupta et al., 2013, Chalermsook et al., 2024].

For an overview of the LP methods for Sparsest Cut, see [Chekuri, 2024].

1.3 open problems

One major open problem for Sparsest Cut is the best approximation ratio for planar graphs. It is conjectured that the ratio for planar graphs is $O(1)$ but currently the best lowerbound is $O(\sqrt{\log n})$. For graphs treewidth k, an open problem is that whether there is a 2 approximation algorithm that runs in $2^{O(k)} \text{poly}(n)$.

2 Literature Review

The seminal work of Leighton and Rao [1999] starts this line of research. They studied multicommodity flow problem and proved a $O(\log n)$ flow-cut gap. They also developed $O(\log n)$ approximation algorithm for multicommodity flow problems, which can imply $O(\log n)$ approximation for Sparsest Cut and $O(\log^2 n)$ approximation for Non-Uniform Sparsest Cut. The technique is called region growing. They also discovered a lowerbound of $\Omega(\log n)$ via expanders. Note that any algorithm achieving the $O(\log n)$ flow cut gap implies an $O(\log^2 n)$ approximation for Non-Uniform Sparsest Cut, but it is possible to approximate (non-uniform) Sparsest Cut with better ratio. This paper showed that $O(\log^2 n)$ is the best ratio we can achieve using flow-cut gap.

For Non-Uniform Sparsest Cut [Leighton and Rao, 1999] only guarantees a $O(\log^2 n)$ approximation. This is further improved by [Linial et al., 1995] and [Aumann and Rabani, 1998]. Aumann and Rabani [1998] applied metric embedding to Non-Uniform Sparsest Cut and obtained a $O(\log n)$ approximation. The connections between metric embedding and Non-Uniform Sparsest Cut is influential. Non-Uniform Sparsest Cut can be formulated as an integer program. Aumann and Rabani considered the metric relaxation of the IP. They observed that Non-Uniform Sparsest Cut is polynomial time solvable for trees and more generally for all ℓ_1 metrics. The $O(\log n)$ approximation follows from the $O(\log n)$ distortion in the metric embedding theorem.

[Arora et al., 2004] and [Arora et al., 2010] further improved the approximation ratio for Sparsest Cut to $O(\sqrt{\log n})$ via semidefinite relaxation. This is currently the best approximation ratio for Sparsest Cut.

3 The Research Design

4 Time Table

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