Connectivity Interdiction & Perturbed Graphic Matroid Cogirth

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Connectivity interdiction

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Problem (connectivity interdiction [Zenklusen, ORL'14])

Let $G = (V, E)$ be a graph with edge weights $w : E \to \mathbb{Z}_+$ and edge removal costs $c : E \to \mathbb{Z}_+$ and let $B \in \mathbb{Z}_+$ be the budget.
Find $F \subset E$ with $c(F) \leq B$ s.t. the edge connectivity in $G - F$ is minimized.

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Problem (B-free min-cut)

Given the same input, define weight function for cuts

$$w'(\delta(X)) = \min_{\substack{F \subset \delta(X) \\ c(F) \leq B}} w(\delta(X) - F)$$

Find the cut with minimum weight w'.

- drag delivery interdiction
- nuclear smuggling interdiction
- hospital infection control
- ...

Previous works

Unit cost $c(\cdot) = 1$	General case	Random?	Reference
$O(m^2 n^4 \log n)$	$O(m^{2+1/\varepsilon} n^4 \log n)$	$\times$	[Zenklusen, ORL'14]
$\tilde{O}(m + n^4 B)$	$O(n^4 \log(Bw_{max})T)$	$\times$	[Huang *et al.*, IPCO'24]
$\tilde{O}(m^2 + n^3 B)$	$\tilde{O}(m^2 + n^3 T)$	$\times$	this work
$O(mn^4 \log^2 n)$	$O(m^{1+1/\varepsilon} n^4 \log^2 n)$	$\checkmark$	[Zenklusen, ORL'14]
$\tilde{O}(mn^3 \log w_{max})$		$\checkmark$	[Drange *et al.*, AAAI'26]
$\tilde{O}(m + n^3 B)$	$\tilde{O}(m + n^3 T)$	$\checkmark$	this work

Table 1: PTASes for connectivity interdiction

T is the complexity of FPTAS for 0-1 knapsack.

Problem (*normalized min-cut* [Chalermsook *et al.*, ICALP'22])

Given the same input as connectivity interdiction, find

$$\underset{cut\ C, F \subset C}{\arg\min} \frac{w(C - F)}{B - c(F) + 1} \quad s.t. \quad 0 \leq c(F) \leq B.$$

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[1]minimum k-edge connected spanning subgraph

Method in [Huang *et al.*, IPCO'24]

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$$\underset{cut\ C, F \subset C}{\arg\min} \frac{w(C - F)}{B - c(F) + 1} \quad s.t. \quad 0 \le c(F) \le B.$$

... first appear in [Chalermsook *et al.*, ICALP'22] as a subproblem in MWU framework when solving some positive covering LP[1].

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Lemma

There is an edge weight w_τ such that
C^* is a $(2+2\varepsilon)$-approximate min-cut in (G, w_τ), where $w_\tau(e) = \min(\tau c(e), w(e))$ is the truncated edge weight.

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> enumerate approx solutions to normalized min-cut $\log_{1+\varepsilon}(Bnw_{max})$
>   reweight the graph $O(m)$
>   enumerate all $2 + 2\varepsilon$ min-cuts $\tilde{O}(n^4)$
>   run FPTAS for knapsack on the cut

LP method

$$\min \quad \sum_e x_e w(e)$$

$$s.t. \quad \sum_{e \in T} x_e + y_e \geq 1 \qquad \forall \text{spanning tree } T \quad (x + y \text{ is a cut})$$

$$\sum_e y_e c(e) \leq B \qquad (\text{budget for } F)$$

$$y_e, x_e \in \{0, 1\} \quad \forall e$$

The integral gap is $+\infty$!

LP method

Consider its ~~linear relaxation~~ Lagrangian dual.

$$LD = \max_{\lambda \geq 0} \min_{\text{cut } C \text{ and } F \subset C} w(C - F) - \lambda(B - c(F))$$

We are interested in $L(\lambda) = \min_{\text{cut } C \text{ and } F \subset C} w(C) - w(F) + \lambda c(F)$.

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Let C^* be the optimal B-free mincut and let λ^* be the optimal solution to LD.

Lemma

$$L(\lambda^*) \leq w_{\lambda^*}(C^*) < 2L(\lambda^*)$$

Cogirth of perturbed graphic matroids

Matroid

A *matroid* $M = (E, \mathcal{B})$ is a structure on set E.
"Bases" $\mathcal{B}$ is a collection of subsets with the following properties:

- $\mathcal{B} \neq \varnothing$;
- If A and B are distinct members of $\mathcal{B}$ and $a \in A - B$,
  then there exists $b \in B - A$ such that $A - a + b \in \mathcal{B}$.

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$X \subset E$ is a *cocycle* if $X \cap B$ is not empty for all $B \in \mathcal{B}$.
The size of minimum cocycle is the *cogirth*.

Examples

- Graphic matroids. E is the edge set. $\mathcal{B}$ is the collection of all spanning forests. Cogirth is the size of min-cut.
- Uniform matroids. E is a large set. $\mathcal{B}$ is the collection of all subsets of E with 5 elements. Cogirth is $|E| - 4$.
- Binary matroids. E is a set of binary vectors. $\mathcal{B}$ is the collection of maximum linearly independent sets. What is the cogirth?
- ...

Graphic matroids $\subset$ Regular matroids $\subset$ MFMC matroids $\subset$ Binary matroids
P $\qquad$ P $\qquad$ P $\qquad$ NP-Hard

Computing (co)girth

Graphic matroids $\subset$ Regular matroids $\subset$ MFMC matroids $\subset$ Binary matroids
P P P NP-Hard

Conjecture ([Geelen *et al.*, Ann. Comb. 2015])

For any proper minor-closed class $\mathcal{M}$ of binary matroids, there is a polynomial time algorithm for computing the girth of matroids in $\mathcal{M}$.

Perturbed graphic matroid

Theorem ([Geelen *et al.*, Ann. Comb. 2015])

For any proper minor-closed class $\mathcal{M}$ of binary matroids, there exists two constants $k, t \in \mathbb{Z}_+$ such that, for each vertically k-connected matroid $M \in \mathcal{M}$, there exist matrices $A, P \in \mathbb{F}_2^{r \times n}$ such that A is the incidence matrix of a graph, $\mathrm{rank}(P) \leq t$, and either M or M^* is isomorphic to $M(A + P)$.

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Theorem ([Geelen & Kapadia, Combinatorica'17])

There are polynomial-time randomized algorithms for computing the girth and the cogirth of $M(A + P)$.

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We solve the cogirth part.

Geelen & Kapadia reduce the cogirth problem of PGMs to binary matroids $M(A)$ with the following representation,

$$A = \begin{array}{c} \\ V(G) \\ \{s\} \end{array} \overset{\begin{array}{cc} E(G) & T \end{array}}{\begin{pmatrix} A(G) & B \\ C & D \end{pmatrix}} \in \mathbb{F}_2^{(V(G)+s)\times(E(G)\cup T)}$$

where $A(G)$ is the incidence matrix of a graph G, T indexes t new columns and $\{s\}$ indexes one additional row.

$M(A)$ is a $(1, t)$-*signed graft*.

Results on computing the cogirth of $(1, t)$-signed grafts:

- $O(r^5 n)$ random algorithm [Geelen & Kapadia, Combinatorica'17]
- $n^{O(1)}$ deterministic algorithm when the $\{s\}$-indexed row is all 0 [Nägele *et al.*, SODA'18]

Method

Theorem ([Chekuri *et al.*, SOSA'19])

Given a matroid M, let $\lambda(M)$ be its cogirth and let $\sigma(M)$ be the fractional base packing number. If there is some constant c that $\frac{\lambda(M)}{\sigma(M)} < c$, then the cogirth of M can be computed deterministically in polynomial number of calls to the independence oracle.

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Theorem

If M is a $(1, t)$-signed graft, then $\frac{\lambda(M)}{\sigma(M)}$ is $O(2^t)$.

Lemma

Let $M(B)$ be a binary matroid with constant ratio $\frac{\lambda(M)}{\sigma(M)}$.

The ratio $\frac{\lambda(M')}{\sigma(M')}$ is also constant for $M' = M\left(\begin{bmatrix} B \\ \sigma \end{bmatrix}\right)$

Proof sketch of constant ratio

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Lemma

Let M be a binary matroid with representation $A \in \mathbb{F}_2^{n \times m}$.
Let A' be the binary matrix $[A, \tau]$ for any binary vector $\tau \in \mathbb{F}_2^n$.
If M and deletion minors of M have constant gap, then $M(A')/\tau$ has constant gap.