1 Better Distortion with Distribution

There is a well known lowerbound for the distortion of embedding a metric space (X, d) into ℓ_1.

Theorem 1.1 *For any metric space (X, d) on n points, one has*

$$(X, d) \xhookrightarrow{\Omega(\log n)} \ell_1.$$

For ℓ_2 the lowerbound is still $\Omega(\log n)$ [1].

Recall that we want to find a $(O(k), (1+\varepsilon)c)$-outlier embedding into ℓ_2 for any metric space (X, d) which admits a (k, c)-outlier embedding into ℓ_2. If we can do this deterministically, we actually find an embedding of the outlier points into ℓ_2 with distortion $O(k)$, which contradicts the lowerbound. However, maybe we can do $O(k)$ via embedding into some distribution of ℓ_2 metrics.

Let (X, d) be a finite metric space and let $\mathcal{Y} = \{(Y_1, d_1), \ldots (Y_h, d_h)\}$ be a set of metric spaces. Let π be a distribution on $\mathcal{Y}$. The original metric space (X, d) embeds into π with distortion D if there is an $r > 0$ such that for all $x, y \in X$,

$$r \leq \frac{\mathrm{E}_{i \leftarrow \pi}[d_i(\alpha_i(x), \alpha_i(y))]}{d(x, y)} \leq Dr.$$

SODA23 paper also embeds (X, d) into distribution.

1.1 Example: Random Trees

Consider the problem of embedding some finite metric into a tree metric. We can get an $O(n)$ lowerbound via the unit edge length cycle C_n. However, if embedding into distortions is allowed, we can do $O(\log n)$.

Theorem 1.2 (Bartal) *Let (X, d) be a metric space on n points with diameter Δ, let $\mathcal{D}T$ be the set of tree metrics that dominate d, there is a distribution π on $\mathcal{D}T$ such that (X, d) embeds into pi with distortion $O(\log n \log \Delta)$.*

[1] https://web.stanford.edu/class/cs369m/cs369mlecture1.pdf