Large-Scale Trade-Off Curve Computation for Incentive Allocation with Cardinality and Matroid Constraints

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Incentive allocation with constraints

A ride sharing company wants to send riders promotional coupons in the hope of more rides.

Image courtesy: ChatGPT-5

Multiple-choice knapsack

Input: n sets of coupons $K_1, \dots, K_n$. Each coupon $e \in K_i$ has a non-negative cost $c_e \in \mathbb{Z}_+$ and value $v_e \in \mathbb{Z}_+$. A positive budget $b \in \mathbb{Z}_+$.
Output: A subset of coupons K that maximizes the total value $\sum_{e \in K} v_e$ while satisfying $|K \cap K_i| \leq 1$ and $\sum_{e \in K} c_e \leq b$.

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Three problems with this formulation:

1. Finding the exact optimum is NP-hard. So we consider solving it approximately.
2. Companies may run multiple campaigns at the same time. So a trade-off curve between budget and profit will be useful.
3. The multiple-choice constraint $|K \cap K_i| \leq 1$ is too strong for real-world applications.

Linear programming relaxation

Input: n sets of coupons $K_1, \ldots, K_n$. Each coupon $e \in K_i$ has a non-negative cost $c_e \in \mathbb{Z}_+$ and value $v_e \in \mathbb{Z}_+$. ~~A positive budget $b \in \mathbb{Z}_+$.~~

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$$
\begin{aligned}
\tau(b) = \max_x \quad & v \cdot x \\
s.t. \quad & c \cdot x \leq b \\
& x_{K_i} \in P_{K_i} \quad \forall i \in [n]
\end{aligned}
$$

Output: function $\tau(b)$ for $b \in (0, +\infty)$.

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$$\begin{aligned} \tau(b) = \max_x \quad & v \cdot x \\ s.t. \quad & c \cdot x \leq b \\ & x_{K_i} \in P_{K_i} \quad \forall i \in [n] \end{aligned}$$

Output: function $\tau(b)$ for $b \in (0, +\infty)$.
We focus on 2 kinds of constraints of $x_{K_i} \in P_{K_i}$.

1. Cardinality. $x_{K_i} \in P_{K_i} \to \sum_{e \in K_i} x_e \leq p$.
2. Matroid. $x_{K_i} \in P_{K_i} \to x_{K_i}$ is in the base polytope of a matroid M_i.

Results

We compute the curve $\tau(b)$ fast.

Theorem 1 *Consider an incentive allocation problem with a total of m incentives. The trade-off curve is a piecewise linear concave function with k breakpoints.*

- *Cardinality constraint.* $k = O(mp^{1/3})$ *and* τ *can be computed in* $O((k + m)\log m)$ *time.*
- *Matroid constraint.* $k = O(mr^{1/3})$ *and* τ *can be computed in* $O(Tk + k\log m)$ *time.*

Let's discuss this in detail at my poster!

#2001 Large-Scale Trade-Off Curve Computation for Incentive Allocation with Cardinality and Matroid Constraints

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Problem

We consider the incentive allocation problem with additional constraints.

Input: A set of coupons $E = \biguplus_i E_i$, where each coupon $e \in E$ has a value and a cost $v_e, c_e \in \mathbb{Z}_+$. Budget $B \in \mathbb{Z}_+$. Constraints $\mathcal{F}_i$ on each subset E_i.

Output: A subset $X \subset E$ of coupons that maximizes the total value $\sum_{e\in X} v_e$ while satisfying the budget constraint $\sum_{e\in X} c_e \leq B$ and additional constraints $X \cap E_i \in \mathcal{F}_i$.

This problem is NP-hard. Consider its LP relaxation.

$$\tau(B) = \max_x \; v \cdot x$$

$$\text{s.t.} \; c \cdot x \leq B$$

Signature Function. Let $f_i(\lambda) = \max\{(v_{E_i} - \lambda c_{E_i})x | x \in \text{conv}(\mathcal{F}_i)\}$ be the signature function of agent i. The signature function is piecewise-linar and convex.

Lagrangian Dual. The Lagrangian dual of LP1 is therefore

$$\min_\lambda \left(B\lambda + \sum_i f_i(\lambda) \right).$$

Theorem 4 *$\tau(B)$ is piecewise-linear and concave.*

Computing $\tau(B)$ is straightforward if $f_i(\lambda)$ is known.

Finding $f_i(\lambda)$

Cardinality constraint. For fixed λ, computing $f_i(\lambda) = \max\{(v_{E_i} - \lambda c_{E_i})x \mid \mathbf{1} \cdot x \leq p\}$ is the same as finding the p largest coupons with respect to the weights $v_e - \lambda c_e$. If