#2001 Large-Scale Trade-Off Curve Computation for Incentive Allocation with Cardinality and Matroid Constraints

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Problem

We consider the incentive allocation problem with additional constraints.

Input: A set of coupons $E = \biguplus_i E_i$, where each coupon $e \in E$ has a value and a cost $v_e, c_e \in \mathbb{Z}_+$. Budget $B \in \mathbb{Z}_+$. Constraints $\mathcal{F}_i$ on each subset E_i.

Output: A subset $X \subset E$ of coupons that maximizes the total value $\sum_{e\in X} v_e$ while satisfying the budget constraint $\sum_{e\in X} c_e \leq B$ and additional constraints $X \cap E_i \in \mathcal{F}_i$.

This problem is NP-hard. Consider its LP relaxation.

$$\begin{aligned} \tau(B) = \max_x \ & v \cdot x \\ s.t. \ & c \cdot x \leq B \\ & x_{E_i} \in \operatorname{conv}(\mathcal{F}_i) \quad \forall i \in [n] \\ & x \in [0,1]^m \end{aligned}$$

Output: The entire curve $\tau(B)$ for $B \in [0, \infty)$.

We consider 3 cases of additional constraints $x_{E_i} \in \mathcal{F}_i$:

1. Multiple-choice. $\sum_{e\in E_i} x_e \leq 1$;
2. Cardinality. $\sum_{e\in E_i} x_e \leq p$;
3. Matroid. $x_{E_i} \in$ independence polytope of a matroid.

Existing works & Comparison

Constraint Type	Result	Fixed budget	Trade-off curve
Multiple Choice	Dyer [1984], Zemel [1984]	$O(m)$	-
	Javaudin et al. [2022]	-	$O(m \log m)$
	this paper	-	$O(m \log m)$
Cardinality	Pisinger [2001]	$O(m \log VC)$	-
	Pisinger [2001]	$O(mp + nB)$	-
	Tokuyama [2001]	$O(m \log m)$	-
	this paper	-	$O((k+m)\log m)$
Matroid	Camerini and Vercellis [1984]	$O(m^2 + T \log m)$	-
	Tokuyama [2001]	$O(T \log m)$	-
	this paper	-	$O(Tk + k \log m)$

Table 1: Comparison of algorithms for incentive allocation: m is the total number of incentives, M is the maximum number of incentives over each agent, p is the max rank of the matroid constraint over each agent, or the limit in the cardinality constraint. V and C is the maximum value and cost of the incentives, respectively. B is the budget. $k = O(mp^{1/3})$ is the number of breakpoints of the trade-off curve. T is the time complexity of matroid optimum base algorithm.

Methods

The idea is to take advantage of the independence among the constraints $\mathcal{F}_i$ and reduce the optimization problem to one in computational geometry.

Signature Function. Let $f_i(\lambda) = \max\{(v_{E_i} - \lambda c_{E_i})x \mid x \in \operatorname{conv}(\mathcal{F}_i)\}$ be the signature function of agent i. The signature function is piecewise-linar and convex.

Lagrangian Dual. The Lagrangian dual of LP1 is therefore

$$\min_\lambda \left(B\lambda + \sum_i f_i(\lambda) \right).$$

Theorem 4 *$\tau(B)$ is piecewise-linear and concave.*

Computing $\tau(B)$ is straightforward if $f_i(\lambda)$ is known.

Finding $f_i(\lambda)$

Cardinality constraint. For fixed λ, computing $f_i(\lambda) = \max\{(v_{E_i} - \lambda c_{E_i})x \mid \mathbf{1} \cdot x \leq p\}$ is the same as finding the p largest coupons with respect to the weights $v_e - \lambda c_e$. If λ is not fixed, this is computing the k-*level* of univariate linear functions.

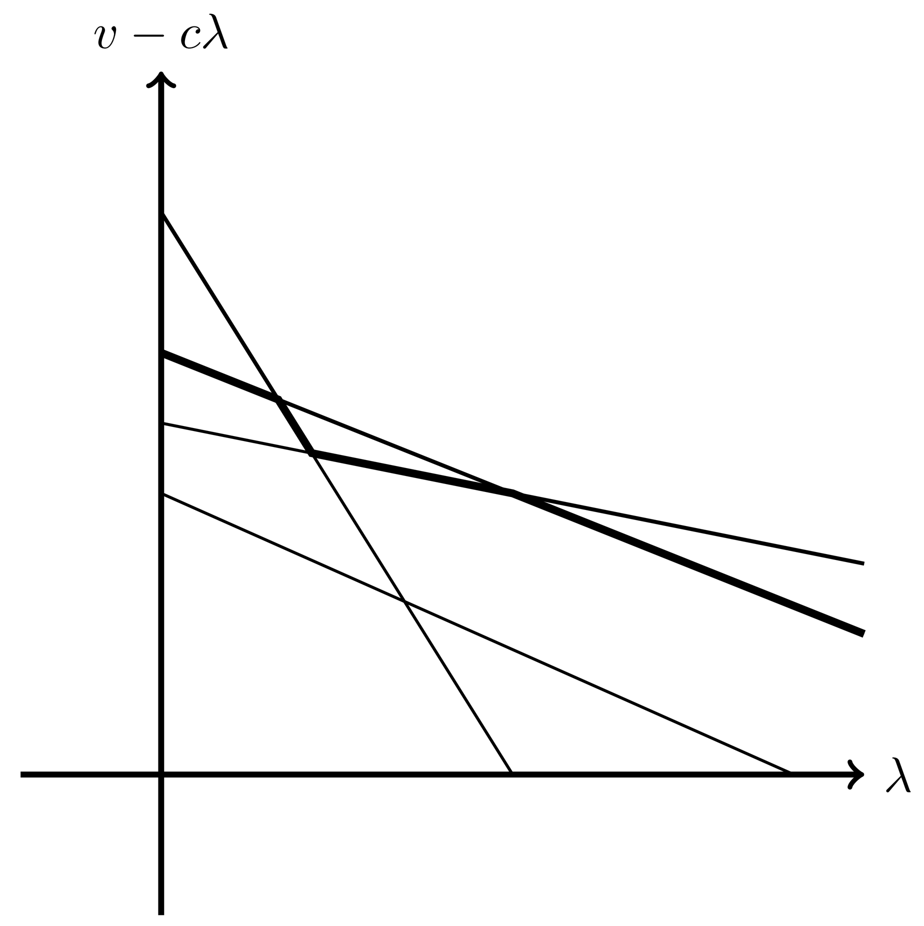

Figure 1: The bold line forms a 2-level in the line arrangement.

Matroid constraint. For fixed λ under matroid constraints, computing the signature function is equivalent to finding the optimum-weight base in a matroid. However, the matroid generalization of k-level problem is significantly harder. We use Eisner-Severance method to compute the curve.

Computational results

m	$p = 20$		$p = 40$		$p = 2000$		$p = m/5$	
	scan	opt	scan	opt	scan	opt	scan	opt
1×10^3	0.000	0.000	0.000	0.001	-	-	0.003	0.002
5×10^3	0.003	0.005	0.006	0.005	0.137	0.027	0.091	0.02
1×10^4	0.008	0.010	0.014	0.012	0.384	0.048	0.384	0.048
5×10^4	0.043	0.089	0.080	0.087	2.634	0.187	9.531	0.326
1×10^5	0.094	0.216	0.173	0.223	5.795	0.397	38.275	1.222
5×10^5	0.528	2.911	0.937	2.952	33.760	3.398	TLE	10.500
1×10^6	1.147	7.291	1.989	7.140	72.485	7.604	TLE	23.203
1×10^7	12.994	100.512	23.863	101.675	TLE	101.775	TLE	133.974

Table 2: The time (in seconds) to compute the breakpoints on the signature function under cardinality constraint using the optimum p-level algorithm (opt) and the scan line algorithm (scan).