Cogirth of Perturbed Graphic Matroids

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1 Introduction

Geelen and Kapadia design randomized polynomial time algorithms for computing girth and cogirth of perturbed graphic matroids [1]. They leave an open problem for the existence of deterministic polynomial time algorithms. We solve the cogirth part using base packing techniques.

A binary matroid is a low rank perturbed graphic matroid (PGM) if it has a binary representation $A+P$, where A is the incidence matrix of a graph and P is a binary matrix with rank at most a constant r. For fixed r, low rank perturbed graphic matroids are closed under minors. PGMs play a central role in the following conjecture.[1]

Conjecture 1 *For any proper minor closed class $\mathcal{M}$ of binary matroids, there is a polynomial time algorithm for computing the girth of matroids in $\mathcal{M}$.*

Geelen and Kapadia reduce the cogirth problem on low rank PGMs to the one on $(1,t)$-signed-grafts. Let G be a graph and let $A(G)$ be its incidence matrix. The incidence matrix of a $(1,t)$-signed-graft $(G,\{s\},T,B,C,D)$ is the binary matrix

$$A = \begin{array}{c} \\ V(G) \\ \{s\} \end{array} \begin{array}{c} \begin{array}{cc} E(G) & T \end{array} \\ \begin{pmatrix} A(G) & B \\ C & D \end{pmatrix} \end{array},$$

where T indexes t new columns and $\{s\}$ indexes a new row. The matroid $M(A)$ is the binary matroid on the matrix $A \in \mathbb{F}_2^{(V(G)+S)\times(E(G)+T)}$.

1.1 previous works

The cogirth problem on $(1,t)$-signed-grafts can be considered as a variation of graph min-cut under congruency constraints.

Problem 1 (t-dimensional even cut, [1]) *Let $G=(V,E)$ be a graph and let $\ell : V \to \mathbb{F}_2^t$ be a t-dimensional coloring on vertices. Given a edge set $C \subseteq E$ and a coloring $D \in \mathbb{F}_2^{1\times t}$, find a non-empty vertex set $X \subseteq V$ that minimizes the smaller value of the following two:*

1. *the minimum $|\delta(X)|$ such that $\sum_{v\in X} \ell(v) = 0$;*
2. *the minimum $|\delta(X) \triangle C|$ such that $\sum_{v\in X} \ell(v) = D$.*

[1] Details can be found in the introduction of [1].

Consider a special case of Problem 1 that $\ell = D = 0$ for all vectices and $C = E$. Then the $\delta(X)$ achieving the minimum value of case 2 is exactly the max-cut of G. This observation suggests that one cannot solve the two cases separately in polynomial time.

Another interesting special case is that $C = \emptyset$ and $D = 0$. The problem becomes graph min-cut with congruency constrants, which is a special case of submodular function minimization under congruency constraints (SFMC) studied by Nägele *et al.* [3]. They show that SFMC with constant number of constraints can be solved deterministically in polynomial time if the modular is prime. However, the objective in case 2 of Problem 1 is not submodular so their method does not generalize.

1.2 proof outline

We prove that if $M(A)$ is a binary matroid on $(1,t)$-signed-graft and t is a constant then $M(A)/T$ has constant cogirth-packing gap, thus showing a deterministic polynomial-time algorithm for computing the cogirth of low rank PGMs.

We first prove that for any binary matroid with constant gap, adding another row to it still keeps the gap constant. Then we can work on the incidence matrix of graphs with t additional columns.

Contracting a set T from a matroid M can be done by contracting elements in T one by one, $M/e_1/\ldots/e_t$. We fix an arbitrary order for elements in T and let T_i be the set of first i elements. We prove that if $(M/T_i)|E(G)$ has constant gap then so does $(M/T_{i+1})|E(G)$.

2 Proof of constant gap

notations and useful lemmas Let $M = (E, \mathcal{I})$ be a matroid. We write $\lambda(M)$ for the cogirth of M and write $\sigma(M)$ for the strength of M. It is known that the strength of M equals the maximum fractional base packing number and $\sigma(M) = \min_{F \subseteq E} \frac{|E-F|}{r(E)-r(F)}$.

In the proof we frequently use the following result on binary matroids. The *cocircuit space* of a binary matroid $M = (E, \mathcal{I})$ is the subspace of $\mathbb{F}_2^E$ that are generated by the incidence vectors of cocircuits of M.

Lemma 2 ([4], Chapter 9) *Let $M = (E, \mathcal{I})$ be a binary matroid and let $A \in \mathbb{F}_2^{r(M)\times|E|}$ be its binary representation. Then the row space of A equals the cocircuit space of M. Any non-zero vector in the cocircuit space represents a dependent set of M.*

We first show that adding one additional row keeps the gap constant.

Lemma 3 *Let M be a binary matroid with binary representation $B \in \mathbb{F}_2^{n\times m}$. If M has constant gap, then $M\left(\begin{bmatrix} B \\ \sigma \end{bmatrix}\right)$ has constant gap for any row vector $\sigma \in \mathbb{F}_2^m$.*

Proof: Let M' be $M\left(\begin{bmatrix} B \\ \sigma \end{bmatrix}\right)$ and let the ground set be E. We have $\lambda(M') \le \lambda(M)$ since the row space becomes larger.

Let $F^* \in \arg\min_{F \subseteq E} \frac{|E-F|}{r'(E)-r'(F)}$. Let r be the rank of M and let r' be the rank of M'. We can assume that $r(E) - r(F^*) \ge 1$ since otherwise we have $r'(E) - r'(F^*) \le 1$ which implies the gap of M' is 1.

$$\begin{aligned}\sigma(M') = \frac{|E-F^*|}{r'(E)-r'(F^*)} &\ge \frac{|E-F^*|}{r(E)+1-r(F^*)} \\ &\ge \frac{|E-F^*|}{2(r(E)-r(F^*))} \ge \frac{\sigma(M)}{2}\end{aligned}$$

Thus we have $\frac{\lambda(M')}{\sigma(M')} \leq \frac{2\lambda(M)}{\sigma(M)}$. $\square$

Now we show that if $(M/T_i)|E(G)$ and its deletion minors have constant gap then $(M/T_{i+1})|E(G)$ also has constant gap.

Lemma 4 *Let M be a binary matroid with representation $A \in \mathbb{F}_2^{n\times m}$. Let A' be the binary matrix $[A, \tau]$ for any binary vector $\tau \in \mathbb{F}_2^n$. If M and deletion minors of M have constant gap, then $M(A')/\tau$ has constant gap.*

Graphic case Before proving this lemma consider an easier case where M is graphic. The new element τ identifies a vertex set T. Note that for a cut $\delta(X)$ one can sum up the rows labelled by $v \in X$ in the incidence matrix and get an indicator vector of $\delta(X)$. A cut $\delta(X)$ is even if $|X \cap T|$ is even. Then the minimum cocircuit of $M(A')/\tau$ is the minimum even cut in (G, T). Let $G = (V, E)$ be a graph and let $T \subseteq V$ be a vertex set with even size. An edge set $F \subseteq E$ is a *T-join* if X induces a subgraph in which the set of vertices with odd degree is exactly T. An cut $\delta(X)$ is a *T-cut* if $|X \cap T|$ odd. Note that any T-cut hits every T-join. For proofs, we refer to Section 12.4 in [2].

For a spanning tree B, let $C(B, \tau)$ be the fundamental circuit in $B \cup \{\tau\}$. Then $C(B, \tau) \cap E$ is a T-join, since it induces a subgraph in which the set of vertices with odd degree is exactly T. One can see that the set of bases is $\{B - e\}$ for all spanning tree B and all $e \in C(B, \tau) \cap B$.

To show that gap we prove the followings,

$$\lambda(M(A')/\tau) \leq \lambda_2(G) \leq 2\sigma_2(G) \leq 2(|J| + \sigma(M(A')/\tau)|_{E\setminus J}),$$

where $\lambda_2(G)$ and $\sigma_2(G)$ are the integral and fractional 3-cut in G, and J is the minimum T-cut.

Any 3-cut in G hits every spanning tree at least twice. So any 3-cut is a hitting set for the bases of $M(A')/\tau$. It follows that $\lambda(M(A')/\tau) \leq \lambda_2(G)$. The second inequality holds since the integrality gap for graph k-cut is 2.

For the last inequality, we claim that $x = \chi_J + \sigma(M(A')/\tau)|_{E\setminus J}$ is a feasible solution to 3-cut LP. First it is easy to see that $x(e) \leq 1$ for any edge e. Consider the size of $J \cap B$ for a spanning tree B. If this is at least 2, then x satisfies the 3-cut constraint for B; Otherwise, $B \setminus J$ should be a base of M'/τ since T-cut intersects every T-join, and we have $\sigma(M'/T)|_{B\setminus J} \geq 1$ as a constraint in the base hitting set LP of $M(A')/\tau$.

To finish the proof consider the connections in the minimum cut of G, the minimum T-cut of G and the minimum even cut of G. Recall that the minimum T-cut is the minimum cut $\delta(X)$ with $|X \cap T|$ odd. Then the minimum cut is either a T-cut or an even cut.

1. If J is the mincut. $\lambda(M'/t) \leq 2(\lambda(G) + \sigma(M'/t)) \leq 2(2\sigma(G) + \sigma(M'/t)) \leq 6\sigma(M'/t)$
2. If $\lambda(M'/t)$ is the mincut. $\lambda(M'/t) = \lambda(G) \leq 2\sigma(G) \leq 2\sigma(M'/t)$

For either case, we have constant gap.

Now we generalize this idea to binary matroids.

Note that we need constant gap for all deletion minors of M since we are going to use the gap of 2-cocycle LP of M.

Lemma 5 ($(1, d)$-good implies (k, kd)-good) ...

Instead of T-cut we find the minimum hitting set of $C(B, \tau)$ for all base B. An useful lemma is the following. We write $M + f$ for an one element extension on the binary representation of M.

Lemma 6 *Let M be a binary matroid and let M' be $(M+f)/f$ for a new element f. The cogirth of M is either the minimum hitting set of $\{C(B,f)\setminus f | \forall B \in \mathcal{B}(M)\}$ or the cogirth of M'.*

Proof: For binary matroid M, finding the cogirth of M is equivalent to finding the smallest nonempty set in the cocircuit space.

Let $A \in \mathbb{F}_2^{r\times n}$ be the binary representation of M. Note that the cocircuit space of M is the same as the row space of A. So we will work on binary matrices. f-extension can be seen as adding a new column v to the binary matrix A. None-empty sets in the cocircuit spaces are called *cocycles*. Any cocycle has an indicator vector yA and thus we can use a row vector y to represent any cocycle. We say a cocycle is even if $yv = 0$ and odd if $yv = 1$.

First consider the cogirth of M'. It follows from definitions that the minimum hitting set of $\{B - e | \forall B \in \mathcal{B}(M), \forall e \in B \cap C(B,f)\}$ is exactly the minimum cocircuit of M', which is the minimum set in the cocircuit space with $yv = 0$. So the cogirth of M' is the minimum even cocycle.

On the other hand, consider a set X which fails to hit all $\{C(B,f)\setminus f | \forall B\}$. Then there must be a set $F \subseteq E \setminus X$ that span the new element f. On matrix, this is equivalent to the existence of a indicator vector χ_F such that $A\chi_F = v$. Now we apply the following lemma which is easy to prove.

Lemma 7 (Farkas' lemma on $\mathbb{F}_2$) *Let A be a $n \times m$ binary matrix and b be a n-dimensional binary vector. Either there is a $x \in \mathbb{F}_2^m$ such that $Ax = b$, or there is $y \in \mathbb{F}_2^n$ such that $y^T A = 0$ and $y^T b = 1$.*

Then X fails to be a hitting set is equivalent to the fact that there does not exist y satisfying $\mathrm{supp}(yA) \subseteq X \wedge yv = 1$. So X hits all $\{C(B,f)\setminus f | \forall B\}$ iff there is such a y. Minimizing X pushes it to $\mathrm{supp}(yA)$ which is an odd cocycle. Hence, the minimum hitting set for $\{C(B,f)\setminus f | \forall B\}$ is the minimum odd cocycle.

The theorem then follows directly from the fact that any cocycle is either odd or even. □

Now we prove Lemma 4.

Proof: We follow the same framework as the graphic case. Let c be the 2-hitting set gap and let c' be the cogirth-packing gap of M. We have

$$\lambda(M'/\tau) \le \lambda_2(M) \le c\sigma_2(M) \le c(|J| + \sigma(M'/T)|_{E\setminus J}) \tag{1}$$

where J is the minimum hitting set of $\{C(B,f)\setminus f | \forall B\}$ and $\lambda_2(M)$ ($\sigma_2(M)$) is the integral (fractional) 2-hitting set of bases of M.

We then apply Lemma 6. If $\lambda(M'/\tau) = \lambda(M)$, then we have $\lambda(M'/\tau) = \lambda(M) \le c\sigma(M) \le c\sigma(M'/\tau)$, since the optimal solution to $\sigma(M'/\tau)$ is feasible to $\sigma(M)$; Otherwise, we extend Equation 1 and get $\lambda(M'/\tau) \le c(|J| + \sigma(M'/\tau)) = c(\lambda(M) + \sigma(M'/\tau)) \le c(c'+1)\sigma(M'/\tau)$. □

To finish the proof of constant gap of M/T we note that $(M/T_t)|E(G) = M/T$ and $(M/T_{i+1})|E(G) = [((M/T_i)|E(G)) + e_{i+1}]/e_{i+1}$ and the constant gap for deletion minors can be proven by wrapping the entire proof into an induction on the size of $E(G)$.

References

[1] Jim Geelen and Rohan Kapadia. Computing Girth and Cogirth in Perturbed Graphic Matroids. *Combinatorica*, 38(1):167–191, February 2018.

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[3] Martin Nägele, Benny Sudakov, and Rico Zenklusen. Submodular Minimization Under Congruency Constraints. *Combinatorica*, 39(6):1351–1386, December 2019.

[4] James Oxley. *Matroid Theory*. Oxford University Press, February 2011.