1 "Cut-free" Proof

Problem 1 (b-free knapsack) *Consider a set of elements E and two weights $w : E \to \mathbb{Z}_+$ and $c : E \to \mathbb{Z}_+$ and a budget $b \in \mathbb{Z}_+$. Given a feasible set $\mathcal{F} \subseteq 2^E$, find $\min_{X \in \mathcal{F}, F \subseteq E} w(X \setminus F)$ such that $c(F) \leq b$.*

Always remember that $\mathcal{F}$ is usually not explicitly given.

Problem 2 (Normalized knapsack) *Given the same input as Problem 1, find $\min\limits_{X \in \mathcal{F}, F \subseteq E} \frac{w(X \setminus F)}{b+1-c(F)}$ such that $c(F) \leq b$.*

Denote by τ the optimum of Problem 2. Define a new weight $w_\tau : E \to \mathbb{R}$,

$$w_\tau(e) = \begin{cases} w(e) & \text{if } w(e) < \tau \cdot c(e) \text{ (light elem)} \\ \tau \cdot c(e) & \text{otherwise (heavy elem)} \end{cases}$$

Lemma 1.1 *Let (X^N, F^N) be the optimal solution to Problem 2. Every element in F^N is heavy.*

proof is exactly the same as lemma 1 in [2].

Lemma 1.2 *For any $X \in \mathcal{F}$, $w_\tau(X) \geq \tau(1 + b)$.*

proof is the same.

Lemma 1.3 $X^N \in \arg\min\limits_{X \in \mathcal{F}} w_\tau(X)$.

Proof:

$$\begin{aligned} w_\tau(X^N) &\leq w(X^N \setminus F^N) + w_\tau(F^N) \\ &= \tau \cdot (b + 1 - c(F^N)) + \tau \cdot c(F^N) \\ &= \tau(b + 1) \end{aligned}$$

Thus by Lemma 1.2, X^N gets the minimum. $\square$

Let (X^*, F^*) be the optimal solution to Problem 1.

Lemma 1.4 X^* *is either an α-approximate solution to $\min_{X \in \mathcal{F}} w_\tau(X)$ for some $\alpha > 1$, or $w(X^* \setminus F^*) \geq \tau(\alpha + (\alpha - 1)b)$.*

The proof is the same.
In fact, corollary 1 and theorem 5 are also the same as those in [2]. Finally we get a knapsack version of Theorem 4:

Theorem 1.5 (Theorem 4 in [2]) *Let $X^{\min}$ be the optimal solution to $\min_{X \in \mathcal{F}} w_\tau(X)$. The optimal set X^* in Problem 1 is a 2-approximation to $X^{\min}$.*

Thus to obtain a FPTAS for Problem 1, one need to design a FPTAS for Problem 2 and a polynomial time alg for finding all 2-approximations to $\min_{X \in \mathcal{F}} w_\tau(X)$.

FPTAS for Problem 2 in [2] (The name "FPTAS" here is not precise since we do not have a approximation scheme but an enumeration algorithm. But I will use this term anyway.) In their settings, $\mathcal{F}$ is the collection of all cuts in some graph. Let OPT^N be the optimum of Problem 2. We can assume that there is no $X \in \mathcal{F}$ s.t. $c(X) \leq b$ since this is polynomially detectable (through min-cut on $c(\cdot)$) and the optimum is 0. Thus we have $\frac{1}{b+1} \leq \text{OPT}^N \leq |E| \cdot \max_e w(e)$. Then we enumerate $\frac{(1+\varepsilon)^i}{b+1}$ where $i \in \left\{0, 1, \ldots, \left\lfloor \log_{1+\varepsilon}(|E|w_{\max}(b+1)) \right\rfloor \right\}$. There is a feasible i s.t. $(1-\varepsilon)\text{OPT}^N \leq \frac{(1+\varepsilon)^i}{b+1} \leq \text{OPT}^N$ since $\frac{(1+\varepsilon)^i}{b+1} \leq \text{OPT}^N \leq \frac{(1+\varepsilon)^{i+1}}{b+1}$ holds for some i.

Note that this enumeration scheme also holds for arbitrary $\mathcal{F}$ if we have a non-zero lowerbound on OPT^N.

Conjecture 1.6 *Let (C, F) be the optimal solution to connectivity interdiction. The optimum cut C can be computed in polynomial time. In other words, connectivity interdiction is almost as easy as knapsack.*

2 Connections

For unit costs, connectivity interdiction with budget $b = k-1$ is the same problem as finding the minimum weighted edge set whose removal breaks k-edge connectivity.

It turns out that Problem 2 is just a necessary ingredient for MWU. Authors of [2] $\subseteq$ authors of [1].

How to derive normalized min cut for connectivity interdiction?

3 Random Stuff

3.1 remove box constraints

Given a positive covering LP,

$$LP1 = \min \quad \sum_e w(e)x_e$$
$$s.t. \quad \sum_{e \in T} c(e)x_e \geq k \quad \forall T$$
$$c(e) \geq x_e \geq 0 \quad \forall e,$$

we want to remove constraints $c(e) \geq x_e$. Consider the following LP,

$$LP2 = \min \quad \sum_e w(e)x_e$$
$$s.t. \quad \sum_{e \in T} c(e)x_e \geq k \qquad \forall T$$
$$\sum_{e \in T \setminus f} c(e)x_e \geq k - c(f) \quad \forall T \; \forall f \in T$$
$$x_e \geq 0 \qquad \forall e,$$

These two LPs have the same optimum. One can see that any feasible solution to LP1 is feasible in LP2. Thus $\text{OPT}(LP1) \geq \text{OPT}(LP2)$. Next we show that any x_e in the optimum

solution to LP2 is always in $[0, c(e)]$. Let x^* be the optimum and suppose that $c(f) < x_f \in x^*$. Consider all constraints $\sum_{e \in T \setminus f} c(e)x_e \geq k - c(f)$ on $T \ni f$. For any such constraint, we have $\sum_{e \in T} c(e)x_e > k$ since we assume $x_f > c(f)$, which means we can decrease x_f without violating any constraint. Thus it contradicts the assumption that x^* is optimal. Then we can add redundant constraints $x_e \leq c(e) \ \forall e$ to LP2 and see that LP1 and LP2 have the same optimum.

This applies to [1] but cannot get an improvement on their algorithm.(MWU does not care the number of constraints.) So does this trick apply to connectivity interdiction?

$$\min_{\text{cut } C, f \in C} \frac{\sum_{e \in C \setminus \{f\}} w(e)x_e}{k - c(f)}$$

References

[1] Parinya Chalermsook, Chien-Chung Huang, Danupon Nanongkai, Thatchaphol Saranurak, Pattara Sukprasert, and Sorrachai Yingchareonthawornchai. Approximating k-Edge-Connected Spanning Subgraphs via a Near-Linear Time LP Solver. *LIPIcs, Volume 229, ICALP 2022*, 229:37:1–37:20, 2022.

[2] Chien-Chung Huang, Nidia Obscura Acosta, and Sorrachai Yingchareonthawornchai. An FPTAS for Connectivity Interdiction. In Jens Vygen and Jarosław Byrka, editors, *Integer Programming and Combinatorial Optimization*, volume 14679, pages 210–223, Cham, 2024. Springer Nature Switzerland.