1 "Cut-free" Proof

Problem 1 (b-free knapsack) *Consider a set of elements E and two weights $w : E \to \mathbb{Z}_+$ and $c : E \to Z_+$ and a budget $b \in \mathbb{Z}_+$. Given a feasible set $\mathcal{F} \subseteq 2^E$, find $\min_{X \in \mathcal{F}, F \subseteq E} w(X \setminus F)$ such that $c(F) \le b$.*

Always remember that $\mathcal{F}$ is usually not explicitly given.

Problem 2 (Normalized knapsack) *Given the same input as Problem 1, find $\min\limits_{X \in \mathcal{F}, F \subseteq E} \frac{w(X \setminus F)}{B - c(F)}$ such that $c(F) \le b$.*

In [4] the normalized min-cut problem use $B = b + 1$. Here we use any integer $B > b$ and see how their method works.

Denote by τ the optimum of Problem 2. Define a new weight $w_\tau : \mathrm{E} \to \mathbb{R}$,

$$w_\tau(e) = \begin{cases} w(e) & \text{if } w(e) < \tau \cdot c(e) \text{ (light elem)} \\ \tau \cdot c(e) & \text{otherwise (heavy elem)} \end{cases}$$

Lemma 1.1 *Let (X^N, F^N) be the optimal solution to Problem 2. Every element in F^N is heavy.*

The proof is exactly the same as [4, Lemma 1].

The following two lemmas show (a general version of) that the optimal cut C^N to normalized min-cut is exactly the minimum cut under weights w_τ.

Lemma 1.2 *For any $X \in \mathcal{F}$, $w_\tau(X) \ge \tau B$.*

Lemma 1.3 $X^N \in \arg\min\limits_{X \in \mathcal{F}} w_\tau(X)$.

Proof:

$$\begin{aligned} w_\tau(X^N) &\le w(X^N \setminus F^N) + w_\tau(F^N) \\ &= \tau \cdot (B - c(F^N)) + \tau \cdot c(F^N) \\ &= \tau B \end{aligned}$$

Thus by Lemma 1.2, X^N gets the minimum. □

Now we show the counter part of [4, Theorem 5], which states the optimal solution to Problem 1 is a α-approximate solution to $\min_{F \in \mathcal{F}} w_\tau(F)$.

Lemma 1.4 (Lemma 4 in [4]) *Let (X^*, F^*) be the optimal solution to Problem 1. X^* is either an α-approximate solution to $\min_{X \in \mathcal{F}} w_\tau(X)$ for some $\alpha > 1$, or $w(X^* \setminus F^*) \ge \tau(\alpha B - b)$.*

Then following the argument of Corollary 1 in [4], assume that X^* is not an α-approximate solution to $\min_{X \in \mathcal{F}} w_\tau(X)$ for some $\alpha > 1$. We have

$$\frac{w(C^N \setminus F^N)}{w(C^* \setminus F^*)} \le \frac{\tau(B - c(F^N))}{\tau(\alpha B - b)} \le \frac{B}{\alpha B - b},$$

where the second inequality uses Lemma 1.4. One can see that if $\alpha > 2$, $\frac{w(C^N \setminus F^N)}{w(C^* \setminus F^*)} \le \frac{B}{\alpha B - b} < 1$ which implies (C^*, F^*) is not optimal. Thus for $\alpha > 2$, X^* must be a 2-approximate solution to $\min_{X \in \mathcal{F}} w_\tau(X)$.

Finally we get a knapsack version of Theorem 4:

Theorem 1.5 (Theorem 4 in [4]) *Let $X^{\min}$ be the optimal solution to $\min_{X\in\mathcal{F}} w_\tau(X)$. The optimal set X^* in Problem 1 is a 2-approximation to $X^{\min}$.*

Thus to obtain a FPTAS for Problem 1, one need to design a FPTAS for Problem 2 and a polynomial time alg for finding all 2-approximations to $\min_{X\in\mathcal{F}} w_\tau(X)$.

FPTAS for Problem 2 in [4] (The name "FPTAS" here is not precise since we do not have a approximation scheme but an enumeration algorithm. But I will use this term anyway.) In their settings, $\mathcal{F}$ is the collection of all cuts in some graph. Let OPT^N be the optimum of Problem 2. We can assume that there is no $X \in \mathcal{F}$ s.t. $c(X) \le b$ since this is polynomially detectable (through min-cut on $c(\cdot)$) and the optimum is 0. Thus we have $\frac{1}{b+1} \le \mathrm{OPT}^N \le |E|\cdot\max_e w(e)$. Then we enumerate $\frac{(1+\varepsilon)^i}{b+1}$ where $i \in \left\{0, 1, \dots, \left\lfloor \log_{1+\varepsilon}(|E| w_{\max}(b+1)) \right\rfloor\right\}$. There is a feasible i s.t. $(1-\varepsilon)\mathrm{OPT}^N \le \frac{(1+\varepsilon)^i}{b+1} \le \mathrm{OPT}^N$ since $\frac{(1+\varepsilon)^i}{b+1} \le \mathrm{OPT}^N \le \frac{(1+\varepsilon)^{i+1}}{b+1}$ holds for some i.

Note that this enumeration scheme also holds for arbitrary $\mathcal{F}$ if we have a non-zero lowerbound on OPT^N.

Conjecture 1.6 *Let (C, F) be the optimal solution to connectivity interdiction. The optimum cut C can be computed in polynomial time. In other words, connectivity interdiction is almost as easy as knapsack.*

2 Connections

For unit costs, connectivity interdiction with budget $b = k-1$ is the same problem as finding the minimum weighted edge set whose removal breaks k-edge connectivity.

It turns out that Problem 2 is just a necessary ingredient for MWU. Authors of [4] $\subseteq$ authors of [1].

How to derive normalized min cut for connectivity interdiction?

$$
\begin{aligned}
\max \quad & z \\
s.t. \quad & \sum_e y_e c(e) \le B && \text{(budget for } F\text{)} \\
& \sum_{e\in T} x_e \ge 1 && \forall T \quad (x \text{ forms a cut}) \\
& \sum_e \min(0, x_e - y_e) w(e) \ge z \\
& y_e, x_e \in \{0,1\} && \forall e
\end{aligned}
$$

we can assume that $y_e \le x_e$.

$$
\begin{aligned}
\min \quad & \sum_e (x_e - y_e) w(e) \\
s.t. \quad & \sum_{e\in T} x_e \ge 1 && \forall T \quad (x \text{ forms a cut}) \\
& \sum_e y_e c(e) \le B && \text{(budget for } F\text{)} \\
& x_e \ge y_e && \forall e \quad (F \subseteq C) \\
& y_e, x_e \in \{0,1\} && \forall e
\end{aligned}
$$

Now this LP looks similar to the normalized min-cut problem.
A further reformulation (the new x is $x-y$) gives us the following,

$$\begin{aligned} \min \quad & \sum_e x_e w(e) \\ s.t. \quad & \sum_{e\in T} x_e + y_e \geq 1 && \forall T \quad (x \text{ forms a cut}) \\ & \sum_e y_e c(e) \leq B && (\text{budget for } F) \\ & y_e, x_e \in \{0,1\} \quad \forall e \end{aligned}$$

Note that now this is almost a positive covering LP. Let $L(\lambda) = \min\{w(C\setminus F) - \lambda(b - c(F))|\forall \text{cut } C\ \forall F \subseteq C\}$. Consider the Lagrangian dual,

$$\max_{\lambda\geq 0} L(\lambda) = \max_{\lambda\geq 0} \min\{w(C\setminus F) - \lambda(b-c(F)), \forall \text{cut } C\ \forall F\subseteq C\}$$

We have shown that the budget B in normalized min-cut does not really matter as long as $B > b$. Note that $L(\lambda)$ and the normalized min-cut look similar to the principal sequence of partitions of a graph and the graph strength problem.

2.1 graph strength

Assume that the graph G is connected (otherwise add dummy edges). Given a graph $G = (V,E)$ and a cost function $c : V \to \mathbb{Z}_+$, the strength $\sigma(G)$ is defined as $\sigma(G) = \min_\Pi \frac{c(\delta(\Pi))}{|\Pi|-1}$, where Π is any partition of V, $|\Pi|$ is the number of parts in the partition and $\delta(\Pi)$ is the set of edges between parts. Note that an alternative formulation of strength (using graphic matroid rank) is $\sigma(G) = \min_{F\subseteq E} \frac{|E-F|}{r(E)-r(F)}$, which in general is the fractional optimum of matroid base packing.

The principal sequence of partitions of G is a piecewise linear concave curve $L(\lambda) = \min_\Pi c(\delta(\Pi)) - \lambda|\Pi|$. Cunningham used principal partition to computed graph strength [3]. There is a list of good properties mentioned in [2, Section 6](implicated stated in [3]).

- $L(\lambda)$ is piecewise linear concave since it is the lower envelope of some line arrangement.

- For each line segment on $L(\lambda)$ there is a corresponding partition Π. If λ^* is a breakpoint on $L(\lambda)$, then there are two optimal solution (say partitions P_1 and P_2, assuming $|P_1| \leq |P_2|$) to $\min_\Pi c(\delta(\Pi)) - \lambda^*|\Pi|$. Then P_2 is a refinement of P_1.

  Proof (sketch): Suppose that P_2 is not a refinement of P_1. We claim that the meet of P_1 and P_2 achieves a smaller objective value than P_1 or P_2 does. For simplicity we assume G is connected. The correspondence between graphic matroid rank function and partitions of V gives us a reformulation $L(\lambda^*) = \min_{F\subseteq E} c(E-F) - \lambda^*(r(E) - r(F) + 1)$. Then the claim is equivalent to the fact that for two optimal solutions F_1, F_2 to $L(\lambda^*)$, $F_1 \subseteq F_2$, which can be seen by submodularity of matroid rank functions. □

- Let λ^* be a breakpoint on $L(\lambda)$ induced by edge set F. The next breakpoint is induced by the edge set F' such that F' contains F and $F' - F$ is the solution to strength problem on the smallest strength component of $G \setminus F$.

(there is a ±1 difference between principal partition and graph strength... but we dont care those $c\lambda$ terms since the difficult part is minimize $L(\lambda)$ for fixed λ)

3 Random Stuff

3.1 remove box constraints

Given a positive covering LP,

$$\begin{aligned} LP1 = \min \quad & \sum_{e} w(e)x_e \\ s.t. \quad & \sum_{e\in T} c(e)x_e \geq k \quad \forall T \\ & c(e) \geq x_e \geq 0 \quad \forall e, \end{aligned}$$

we want to remove constraints $c(e) \geq x_e$. Consider the following LP,

$$\begin{aligned} LP2 = \min \quad & \sum_{e} w(e)x_e \\ s.t. \quad & \sum_{e\in T} c(e)x_e \geq k & \forall T \\ & \sum_{e\in T\setminus f} c(e)x_e \geq k - c(f) & \forall T\ \forall f \in T \\ & x_e \geq 0 & \forall e, \end{aligned}$$

These two LPs have the same optimum. One can see that any feasible solution to LP1 is feasible in LP2. Thus $\mathrm{OPT}(LP1) \geq \mathrm{OPT}(LP2)$. Next we show that any x_e in the optimum solution to LP2 is always in $[0, c(e)]$. Let x^* be the optimum and suppose that $c(f) < x_f \in x^*$. Consider all constraints $\sum_{e\in T\setminus f} c(e)x_e \geq k - c(f)$ on $T \ni f$. For any such constraint, we have $\sum_{e\in T} c(e)x_e > k$ since we assume $x_f > c(f)$, which means we can decrease x_f without violating any constraint. Thus it contradicts the assumption that x^* is optimal. Then we can add redundant constraints $x_e \leq c(e)\ \forall e$ to LP2 and see that LP1 and LP2 have the same optimum.

This applies to [1] but cannot get an improvement on their algorithm.(MWU does not care the number of constraints.) So does this trick apply to connectivity interdiction?

$$\min_{\text{cut C}, f\in C} \frac{\sum_{e\in C\setminus\{f\}} w(e)x_e}{k - c(f)}$$

References

[1] Parinya Chalermsook, Chien-Chung Huang, Danupon Nanongkai, Thatchaphol Saranurak, Pattara Sukprasert, and Sorrachai Yingchareonthawornchai. Approximating k-Edge-Connected Spanning Subgraphs via a Near-Linear Time LP Solver. *LIPIcs, Volume 229, ICALP 2022*, 229:37:1–37:20, 2022.

[2] Chandra Chekuri, Kent Quanrud, and Chao Xu. LP Relaxation and Tree Packing for Minimum k-Cut. *SIAM Journal on Discrete Mathematics*, 34(2):1334–1353, January 2020.

[3] William H. Cunningham. Optimal attack and reinforcement of a network. *Journal of the ACM*, 32(3):549–561, July 1985.

[4] Chien-Chung Huang, Nidia Obscura Acosta, and Sorrachai Yingchareonthawornchai. An FPTAS for Connectivity Interdiction. In Jens Vygen and Jarosław Byrka, editors, *Integer Programming and Combinatorial Optimization*, volume 14679, pages 210–223, Cham, 2024. Springer Nature Switzerland.