

# 1 “Cut-free” Proof

**Problem 1 (b-free knapsack)** Consider a set of elements  $E$  and two weights  $w : E \rightarrow \mathbb{Z}_+$  and  $c : E \rightarrow \mathbb{Z}_+$  and a budget  $b \in \mathbb{Z}_+$ . Given a feasible set  $\mathcal{F} \subseteq 2^E$ , find  $\min_{X \in \mathcal{F}, F \subseteq E} w(X \setminus F)$  such that  $c(F) \leq b$ .

Always remember that  $\mathcal{F}$  is usually not explicitly given.

**Problem 2 (Normalized knapsack)** Given the same input as **Problem 1**, find  $\min_{X \in \mathcal{F}, F \subseteq E} \frac{w(X \setminus F)}{b+1-c(F)}$  such that  $c(F) \leq b$ .

Denote by  $\tau$  the optimum of **Problem 2**. Define a new weight  $w_\tau : E \rightarrow \mathbb{R}$ ,

$$w_\tau(e) = \begin{cases} w(e) & \text{if } w(e) < \tau \cdot c(e) \text{ (light elem)} \\ \tau \cdot c(e) & \text{otherwise (heavy elem)} \end{cases}$$

**Lemma 1.1** Let  $(X^N, F^N)$  be the optimal solution to **Problem 2**. Every element in  $F^N$  is heavy.

proof is exactly the same as lemma 1 in [2].

**Lemma 1.2** For any  $X \in \mathcal{F}$ ,  $w_\tau(X) \geq \tau(1+b)$ .

proof is the same.

**Lemma 1.3**  $X^N \in \arg \min_{X \in \mathcal{F}} w_\tau(X)$ .

**Proof:**

$$\begin{aligned} w_\tau(X^N) &\leq w(X^N \setminus F^N) + w_\tau(F^N) \\ &= \tau \cdot (b+1 - c(F^N)) + \tau \cdot c(F^N) \\ &= \tau(b+1) \end{aligned}$$

Thus by **Lemma 1.2**,  $X^N$  gets the minimum. □

Let  $(X^*, F^*)$  be the optimal solution to **Problem 1**.

**Lemma 1.4**  $X^*$  is either an  $\alpha$ -approximate solution to  $\min_{X \in \mathcal{F}} w_\tau(X)$  for some  $\alpha > 1$ , or  $w(X^* \setminus F^*) \geq \tau(\alpha + (\alpha - 1)b)$ .

The proof is the same.

In fact, corollary 1 and theorem 5 are also the same as those in [2]. Finally we get a knapsack version of Theorem 4:

**Theorem 1.5 (Theorem 4 in [2])** Let  $X^{\min}$  be the optimal solution to  $\min_{X \in \mathcal{F}} w_\tau(X)$ . The optimal set  $X^*$  in **Problem 1** is a 2-approximation to  $X^{\min}$ .

Thus to obtain a FPTAS for **Problem 1**, one need to design a FPTAS for **Problem 2** and a polynomial time alg for finding all 2-approximations to  $\min_{X \in \mathcal{F}} w_\tau(X)$ .

**FPTAS for Problem 2 in [2]** (The name “FPTAS” here is not precise since we do not have a approximation scheme but an enumeration algorithm. But I will use this term anyway.) In their settings,  $\mathcal{F}$  is the collection of all cuts in some graph. Let  $\text{OPT}^N$  be the optimum of Problem 2. We can assume that there is no  $X \in \mathcal{F}$  s.t.  $c(X) \leq b$  since this is polynomially detectable (through min-cut on  $c(\cdot)$ ) and the optimum is 0. Thus we have  $\frac{1}{b+1} \leq \text{OPT}^N \leq |E| \cdot \max_e w(e)$ . Then we enumerate  $\frac{(1+\varepsilon)^i}{b+1}$  where  $i \in \{0, 1, \dots, \lceil \log_{1+\varepsilon}(|E|w_{\max}(b+1)) \rceil\}$ . There is a feasible  $i$  s.t.  $(1-\varepsilon)\text{OPT}^N \leq \frac{(1+\varepsilon)^i}{b+1} \leq \text{OPT}^N$  since  $\frac{(1+\varepsilon)^i}{b+1} \leq \text{OPT}^N \leq \frac{(1+\varepsilon)^{i+1}}{b+1}$  holds for some  $i$ .

Note that this enumeration scheme also holds for arbitrary  $\mathcal{F}$  if we have a non-zero lowerbound on  $\text{OPT}^N$ .

**Conjecture 1.6** *Let  $(C, F)$  be the optimal solution to connectivity interdiction. The optimum cut  $C$  can be computed in polynomial time. In other words, connectivity interdiction is almost as easy as knapsack.*

## 2 Connections

For unit costs, connectivity interdiction with budget  $b = k - 1$  is the same problem as finding the minimum weighted edge set whose removal breaks  $k$ -edge connectivity.

It turns out that Problem 2 is just a necessary ingredient for MWU. Authors of [2]  $\subseteq$  authors of [1].

How to derive normalized min cut for connectivity interdiction?

$$\begin{aligned}
& \max && z \\
& \text{s.t.} && \sum_e y_e c(e) \leq B && (\text{budget for } F) \\
& && \sum_{e \in T} x_e \geq 1 && \forall T \quad (x \text{ forms a cut}) \\
& && \sum_e \min(0, x_e - y_e) w(e) \geq z \\
& && y_e, x_e \in \{0, 1\} \quad \forall e
\end{aligned}$$

we can assume that  $y_e \leq x_e$ .

$$\begin{aligned}
& \min && \sum_e (x_e - y_e) w(e) \\
& \text{s.t.} && \sum_{e \in T} x_e \geq 1 && \forall T \quad (x \text{ forms a cut}) \\
& && \sum_e y_e c(e) \leq B && (\text{budget for } F) \\
& && x_e \geq y_e && \forall e \quad (F \subseteq C) \\
& && y_e, x_e \in \{0, 1\} && \forall e
\end{aligned}$$

Now this LP looks similar to the normalized min-cut problem.

A further reformulation (the new  $x$  is  $x - y$ ) gives us the following,

$$\begin{aligned}
\min \quad & \sum_e x_e w(e) \\
\text{s.t.} \quad & \sum_{e \in T} x_e + y_e \geq 1 \quad \forall T \quad (x \text{ forms a cut}) \\
& \sum_e y_e c(e) \leq B \quad (\text{budget for } F) \\
& y_e, x_e \in \{0, 1\} \quad \forall e
\end{aligned}$$

Note that now this is almost a positive covering LP.

### 3 Random Stuff

#### 3.1 remove box constraints

Given a positive covering LP,

$$\begin{aligned}
LP1 = \min \quad & \sum_e w(e)x_e \\
\text{s.t.} \quad & \sum_{e \in T} c(e)x_e \geq k \quad \forall T \\
& c(e) \geq x_e \geq 0 \quad \forall e,
\end{aligned}$$

we want to remove constraints  $c(e) \geq x_e$ . Consider the following LP,

$$\begin{aligned}
LP2 = \min \quad & \sum_e w(e)x_e \\
\text{s.t.} \quad & \sum_{e \in T} c(e)x_e \geq k \quad \forall T \\
& \sum_{e \in T \setminus f} c(e)x_e \geq k - c(f) \quad \forall T \quad \forall f \in T \\
& x_e \geq 0 \quad \forall e,
\end{aligned}$$

These two LPs have the same optimum. One can see that any feasible solution to LP1 is feasible in LP2. Thus  $\text{OPT}(LP1) \geq \text{OPT}(LP2)$ . Next we show that any  $x_e$  in the optimum solution to LP2 is always in  $[0, c(e)]$ . Let  $x^*$  be the optimum and suppose that  $c(f) < x_f \in x^*$ . Consider all constraints  $\sum_{e \in T \setminus f} c(e)x_e \geq k - c(f)$  on  $T \ni f$ . For any such constraint, we have  $\sum_{e \in T} c(e)x_e > k$  since we assume  $x_f > c(f)$ , which means we can decrease  $x_f$  without violating any constraint. Thus it contradicts the assumption that  $x^*$  is optimal. Then we can add redundant constraints  $x_e \leq c(e) \quad \forall e$  to LP2 and see that LP1 and LP2 have the same optimum.

This applies to [1] but cannot get an improvement on their algorithm. (MWU does not care the number of constraints.) So does this trick apply to connectivity interdiction?

$$\min_{\text{cut } C, f \in C} \frac{\sum_{e \in C \setminus \{f\}} w(e)x_e}{k - c(f)}$$

## References

- [1] Parinya Chalermsook, Chien-Chung Huang, Danupon Nanongkai, Thatchaphol Saranurak, Pattara Sukprasert, and Sorrachai Yingchareonthawornchai. Approximating  $k$ -Edge-Connected Spanning Subgraphs via a Near-Linear Time LP Solver. *LIPICs, Volume 229, ICALP 2022*, 229:37:1–37:20, 2022.
- [2] Chien-Chung Huang, Nidia Obscura Acosta, and Sorrachai Yingchareonthawornchai. An FPTAS for Connectivity Interdiction. In Jens Vygen and Jarosław Byrka, editors, *Integer Programming and Combinatorial Optimization*, volume 14679, pages 210–223, Cham, 2024. Springer Nature Switzerland.