

1 “Cut-free” Proof

Problem 1 (b-free knapsack) Consider a set of elements E and two weights $w : E \rightarrow \mathbb{Z}_+$ and $c : E \rightarrow \mathbb{Z}_+$ and a budget $b \in \mathbb{Z}_+$. Given a feasible set $\mathcal{F} \subseteq 2^E$, find $\min_{X \in \mathcal{F}, F \subseteq E} w(X \setminus F)$ such that $c(F) \leq b$.

Always remember that \mathcal{F} is usually not explicitly given.

Problem 2 (Normalized knapsack) Given the same input as **Problem 1**, find $\min_{X \in \mathcal{F}, F \subseteq E} \frac{w(X \setminus F)}{B - c(F)}$ such that $c(F) \leq b$.

In [4] the normalized min-cut problem use $B = b + 1$. Here we use any integer $B > b$ and see how their method works.

Denote by τ the optimum of **Problem 2**. Define a new weight $w_\tau : E \rightarrow \mathbb{R}$,

$$w_\tau(e) = \begin{cases} w(e) & \text{if } w(e) < \tau \cdot c(e) \text{ (light elem)} \\ \tau \cdot c(e) & \text{otherwise (heavy elem)} \end{cases}$$

Lemma 1.1 Let (X^N, F^N) be the optimal solution to **Problem 2**. Every element in F^N is heavy.

The proof is exactly the same as [4, Lemma 1].

The following two lemmas show (a general version of) that the optimal cut C^N to normalized min-cut is exactly the minimum cut under weights w_τ .

Lemma 1.2 For any $X \in \mathcal{F}$, $w_\tau(X) \geq \tau B$.

Lemma 1.3 $X^N \in \arg \min_{X \in \mathcal{F}} w_\tau(X)$.

Proof:

$$\begin{aligned} w_\tau(X^N) &\leq w(X^N \setminus F^N) + w_\tau(F^N) \\ &= \tau \cdot (B - c(F^N)) + \tau \cdot c(F^N) \\ &= \tau B \end{aligned}$$

Thus by **Lemma 1.2**, X^N gets the minimum. □

Now we show the counter part of [4, Theorem 5], which states the optimal solution to **Problem 1** is a α -approximate solution to $\min_{F \in \mathcal{F}} w_\tau(F)$.

Lemma 1.4 (Lemma 4 in [4]) Let (X^*, F^*) be the optimal solution to **Problem 1**. X^* is either an α -approximate solution to $\min_{X \in \mathcal{F}} w_\tau(X)$ for some $\alpha > 1$, or $w(X^* \setminus F^*) \geq \tau(\alpha B - b)$.

Then following the argument of Corollary 1 in [4], assume that X^* is not an α -approximate solution to $\min_{X \in \mathcal{F}} w_\tau(X)$ for some $\alpha > 1$. We have

$$\frac{w(C^N \setminus F^N)}{w(C^* \setminus F^*)} \leq \frac{\tau(B - c(F^N))}{\tau(\alpha B - b)} \leq \frac{B}{\alpha B - b},$$

where the second inequality uses **Lemma 1.4**. One can see that if $\alpha > 2$, $\frac{w(C^N \setminus F^N)}{w(C^* \setminus F^*)} \leq \frac{B}{\alpha B - b} < 1$ which implies (C^*, F^*) is not optimal. Thus for $\alpha > 2$, X^* must be a 2-approximate solution to $\min_{X \in \mathcal{F}} w_\tau(X)$.

Finally we get a knapsack version of Theorem 4:

Theorem 1.5 (Theorem 4 in [4]) Let X^{\min} be the optimal solution to $\min_{X \in \mathcal{F}} w_\tau(X)$. The optimal set X^* in [Problem 1](#) is a 2-approximation to X^{\min} .

Thus to obtain a FPTAS for [Problem 1](#), one need to design a FPTAS for [Problem 2](#) and a polynomial time alg for finding all 2-approximations to $\min_{X \in \mathcal{F}} w_\tau(X)$.

FPTAS for [Problem 2](#) in [4] (The name ‘‘FPTAS’’ here is not precise since we do not have a approximation scheme but an enumeration algorithm. But I will use this term anyway.) In their settings, \mathcal{F} is the collection of all cuts in some graph. Let OPT^N be the optimum of [Problem 2](#). We can assume that there is no $X \in \mathcal{F}$ s.t. $c(X) \leq b$ since this is polynomially detectable (through min-cut on $c(\cdot)$) and the optimum is 0. Thus we have $\frac{1}{b+1} \leq \text{OPT}^N \leq |E| \cdot \max_e w(e)$. Then we enumerate $\frac{(1+\varepsilon)^i}{b+1}$ where $i \in \{0, 1, \dots, \lfloor \log_{1+\varepsilon}(|E|w_{\max}(b+1)) \rfloor\}$. There is a feasible i s.t. $(1-\varepsilon)\text{OPT}^N \leq \frac{(1+\varepsilon)^i}{b+1} \leq \text{OPT}^N$ since $\frac{(1+\varepsilon)^i}{b+1} \leq \text{OPT}^N \leq \frac{(1+\varepsilon)^{i+1}}{b+1}$ holds for some i .

Note that this enumeration scheme also holds for arbitrary \mathcal{F} if we have a non-zero lowerbound on OPT^N .

Conjecture 1.6 Let (C, F) be the optimal solution to connectivity interdiction. The optimum cut C can be computed in polynomial time. In other words, connectivity interdiction is almost as easy as knapsack.

2 Connections

For unit costs, connectivity interdiction with budget $b = k - 1$ is the same problem as finding the minimum weighted edge set whose removal breaks k -edge connectivity.

It turns out that [Problem 2](#) is just a necessary ingredient for MWU. Authors of [\[4\]](#) \subseteq authors of [\[1\]](#).

How to derive normalized min cut for connectivity interdiction?

$$\begin{aligned}
& \max && z \\
& \text{s.t.} && \sum_e y_e c(e) \leq B && (\text{budget for } F) \\
& && \sum_{e \in T} x_e \geq 1 && \forall T \quad (x \text{ forms a cut}) \\
& && \sum_e \min(0, x_e - y_e) w(e) \geq z \\
& && y_e, x_e \in \{0, 1\} \quad \forall e
\end{aligned}$$

we can assume that $y_e \leq x_e$.

$$\begin{aligned}
& \min && \sum_e (x_e - y_e) w(e) \\
& \text{s.t.} && \sum_{e \in T} x_e \geq 1 && \forall T \quad (x \text{ forms a cut}) \\
& && \sum_e y_e c(e) \leq B && (\text{budget for } F) \\
& && x_e \geq y_e && \forall e \quad (F \subseteq C) \\
& && y_e, x_e \in \{0, 1\} && \forall e
\end{aligned}$$

Now this LP looks similar to the normalized min-cut problem.
A further reformulation (the new x is $x - y$) gives us the following,

$$\begin{aligned}
\min \quad & \sum_e x_e w(e) \\
\text{s.t.} \quad & \sum_{e \in T} x_e + y_e \geq 1 \quad \forall T \quad (x \text{ forms a cut}) \\
& \sum_e y_e c(e) \leq B \quad (\text{budget for } F) \\
& y_e, x_e \in \{0, 1\} \quad \forall e
\end{aligned}$$

Note that now this is almost a positive covering LP. Let $L(\lambda) = \min\{w(C \setminus F) - \lambda(b - c(F)) \mid \forall \text{ cut } C \forall F \subseteq C\}$. Consider the Lagrangian dual,

$$\max_{\lambda \geq 0} L(\lambda) = \max_{\lambda \geq 0} \min \{w(C \setminus F) - \lambda(b - c(F)), \forall \text{ cut } C \forall F \subseteq C\}$$

We have shown that the budget B in normalized min-cut does not really matter as long as $B > b$. Note that $L(\lambda)$ and the normalized min-cut look similar to the principal sequence of partitions of a graph and the graph strength problem.

2.1 graph strength

Assume that the graph G is connected (otherwise add dummy edges). Given a graph $G = (V, E)$ and a cost function $c : V \rightarrow \mathbb{Z}_+$, the strength $\sigma(G)$ is defined as $\sigma(G) = \min_{\Pi} \frac{c(\delta(\Pi))}{|\Pi|-1}$, where Π is any partition of V , $|\Pi|$ is the number of parts in the partition and $\delta(\Pi)$ is the set of edges between parts. Note that an alternative formulation of strength (using graphic matroid rank) is $\sigma(G) = \min_{F \subseteq E} \frac{|E-F|}{r(E)-r(F)}$, which in general is the fractional optimum of matroid base packing.

The principal sequence of partitions of G is a piecewise linear concave curve $L(\lambda) = \min_{\Pi} c(\delta(\Pi)) - \lambda|\Pi|$. Cunningham used principal partition to compute graph strength [3]. There is a list of good properties mentioned in [2, Section 6] (implicated stated in [3]).

- $L(\lambda)$ is piecewise linear concave since it is the lower envelope of some line arrangement.
- For each line segment on $L(\lambda)$ there is a corresponding partition Π . If λ^* is a breakpoint on $L(\lambda)$, then there are two optimal solution (say partitions P_1 and P_2 , assuming $|P_1| \leq |P_2|$) to $\min_{\Pi} c(\delta(\Pi)) - \lambda^*|\Pi|$. Then P_2 is a refinement of P_1 .

Proof (sketch): Suppose that P_2 is not a refinement of P_1 . We claim that the meet of P_1 and P_2 achieves a objective value at least no larger than P_1 or P_2 does. For simplicity we assume G is connected. The correspondence between graphic matroid rank function and partitions of V gives us a reformulation $L(\lambda^*) = \min_{F \subseteq E} c(E-F) - \lambda^*(r(E)-r(F)+1)$. Let $g(F) = c(E-F) + \lambda^*r(F) - \lambda^*n$. Then the claim is equivalent to the fact that for two optimal solutions F_1, F_2 to $L(\lambda^*)$, $g(F_1 \cup F_2) \leq g(F_1) = g(F_2) \leq g(F_1 \cap F_2)$, which can be seen by submodularity of matroid rank functions. \square

- Let λ^* be a breakpoint on $L(\lambda)$ induced by edge set F . The next breakpoint is induced by the edge set F' such that F' contains F and $F' - F$ is the solution to strength problem on the smallest strength component of $G \setminus F$.

(there is a ± 1 difference between principal partition and graph strength... but we don't care those $c\lambda$ terms since the difficult part is minimize $L(\lambda)$ for fixed λ)

3 Random Stuff

3.1 remove box constraints

Given a positive covering LP,

$$\begin{aligned} LP1 = \min \quad & \sum_e w(e)x_e \\ \text{s.t.} \quad & \sum_{e \in T} c(e)x_e \geq k \quad \forall T \\ & c(e) \geq x_e \geq 0 \quad \forall e, \end{aligned}$$

we want to remove constraints $c(e) \geq x_e$. Consider the following LP,

$$\begin{aligned} LP2 = \min \quad & \sum_e w(e)x_e \\ \text{s.t.} \quad & \sum_{e \in T} c(e)x_e \geq k \quad \forall T \\ & \sum_{e \in T \setminus f} c(e)x_e \geq k - c(f) \quad \forall T \forall f \in T \\ & x_e \geq 0 \quad \forall e, \end{aligned}$$

These two LPs have the same optimum. One can see that any feasible solution to LP1 is feasible in LP2. Thus $\text{OPT}(LP1) \geq \text{OPT}(LP2)$. Next we show that any x_e in the optimum solution to LP2 is always in $[0, c(e)]$. Let x^* be the optimum and suppose that $c(f) < x_f \in x^*$. Consider all constraints $\sum_{e \in T \setminus f} c(e)x_e \geq k - c(f)$ on $T \ni f$. For any such constraint, we have $\sum_{e \in T} c(e)x_e > k$ since we assume $x_f > c(f)$, which means we can decrease x_f without violating any constraint. Thus it contradicts the assumption that x^* is optimal. Then we can add redundant constraints $x_e \leq c(e) \forall e$ to LP2 and see that LP1 and LP2 have the same optimum.

This applies to [1] but cannot get an improvement on their algorithm. (MWU does not care the number of constraints.) So does this trick apply to connectivity interdiction?

$$\min_{\text{cut } C, f \in C} \frac{\sum_{e \in C \setminus \{f\}} w(e)x_e}{k - c(f)}$$

References

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