Exercises in Sariel Har-Peled's *Geometric Approximation Algorithms*

For errata and more stuff, see https://sarielhp.org/book/
Note that unless specifically stated, we always consider the RAM model.

1 Grid

Ex 1.1 *Let P be a max cardinality point set contained in the d-dimensional unit hypercube such that the smallest distance of point pairs in P is 1. Prove that*

$$\left(\left\lfloor\sqrt{d}\right\rfloor+1\right)^d \le |P| \le \left(\left\lceil\sqrt{d}\right\rceil+1\right)^d.$$

hmm... the first exercise in this book is wrong. See https://sarielhp.org/book/errata.pdf. The stated lowerbound is actually an upperbound.

Proof: We evenly partition the $[0,1]$ interval into $m = \left(\left\lfloor\sqrt{d}\right\rfloor+1\right)$ small segments for each of the d axes. The unit hypercube is partitioned into m^d cells. The length of each cell's diagonal is $\sqrt{\frac{d}{m^2}} < 1$. Thus there is at most one point of P in each cell and there are $\left(\left\lfloor\sqrt{d}\right\rfloor+1\right)^d$ cells.

For lowerbound, one can construct a solution of size 2^d by selecting vertices of the hypercube. For sufficient large d one can find a solution of size $(\sqrt{d}/5)^d$.[1] Let point set P be the optimal solution and let $n = |P|$. We place a d-dimensional unit sphere around each point of P. These n spheres must cover the unit hypercube since otherwise we can add more points into P. Thus one has $n\,\mathrm{vol}(1b^d) \ge 1$.

$$\begin{aligned} n &\ge 1/\,\mathrm{vol}(1b^d) \\ &= \frac{\Gamma(d/2+1)}{\pi^{d/2}} \\ &\ge \sqrt{2\pi/(d/2+1)}\left(\frac{\sqrt{d}}{\sqrt{2e\pi}}\right)^d \end{aligned}$$

The last line is greater than $(\sqrt{d}/5)^d$ for large enough d. □

Ex 1.2 *Compute clustering radius. Let C and P be two given set of points such that $k = |C|$ and $n = |P|$. Define the covering radius of P by C as $r = \max_{p\in P}\min_{c\in C}\|p-c\|$.*

[1] Exercise 1.1 (C) in https://sarielhp.org/book/chapters/min_disk.pdf

1. *find an $O(n + k \log n)$ expected time alg that outputs α such that $\alpha \le r \le 10\alpha$.*
2. *for prescribed $\varepsilon > 0$, find an $O(n + k\varepsilon^{-2} \log n)$ expected time alg that outputs α s.t. $\alpha < r < (1+\epsilon)\alpha$.*

Not in the book

Problem 1 (d-dimensional rectangle stabbing [GIK02]) *Given a set R of n axis-parallel rectangles and a set $\mathcal{H}$ of axis-parallel d dimensional hyperplanes, find the minimum subset of $\mathcal{H}$ such that every rectangle is stabbed by at least one hyperplane in the subset.*

This problem is NP-hard even for the 2D case. There is a LP rounding method which gives a d-approximation for dimension d. Let $K_i \subset \mathcal{H}$ be the set of hyperplanes that are orthogonal to the ith axis. For a rectangle $r \in R$, denote by K_i^r the set of hyperplanes in K_i that stab r. Consider the following LP.

$$\begin{aligned} \min \quad & \sum_{H \in \mathcal{H}} x_H \\ s.t. \quad & \sum_{i \in [d]} \sum_{H \in K_i^r} x_H \ge 1 \quad \forall r \in R \\ & x_H \ge 0 \quad \forall H \in \mathcal{H} \end{aligned}$$

Let $\{x_H^* : H \in \mathcal{H}\}$ be the optimal solution to the above LP. For each r, there must be some $i \in [d]$ such that $\sum_{H \in K_i^r} x_H^* \ge 1/d$. Denote such a set for rectangle r by K_*^r. Suppose that we find a subset $\mathcal{H}^{int} \subset \mathcal{H}$ and define a integral solution $\{y_H = 1\}_{H \in \mathcal{H}^{int}} \cup \{y_H = 0\}_{H \notin \mathcal{H}^{int}}$ such that $\sum_{H \in K_*^r} \ge 1$ for each rectangle r. In other words, we restrict the solution such that every rectangle r is stabbed by hyperplanes in K_*^r.

One nice property of this restriction is that now the problem becomes independent for each dimension. We assign to each rectangle r a dimension i such that $\sum_{H \in K_i^r} x_H^* \ge 1/d$. This assignment indicates a partition $\{R_i\}_{i \in [d]}$ of R. We want to solve the following IP for dimension $i \in [d]$.

$$\begin{aligned} IP_i = \min \quad & \sum_{H \in K_i} x_H \\ s.t. \quad & \sum_{H \in K_i^r} x_H \ge 1 \qquad \forall r \in R_i \\ & x_H \in \{0, 1\} \quad \forall H \in K_i \end{aligned}$$

Another nice property is that the constraint matrix is TUM since one can sort the hyperplanes in K_i by their intersection with the ith axis and see that element 1's locate consecutively in each row in the constraint matrix. Hence, the linear relaxation of IP_i (denoted by LP_i) is integral and we can solve IP_i in polynomial time.

Now we show connections between x^* and solutions of IP_i. Let $x^*|_{K_i}$ be the optimal solution to the rectangle stabbing LP restricted to hyperplanes in K_i. We also have $\sum_{i \in [d]} \mathrm{OPT}(IP_i) \le d \sum_H x_H^*$ since $dx^*|_{K_i}$ is a feasible solution to LP_i. Then the d-integrality gap follows from the fact that the union of optimal solutions to IP_i is a feasible solution to the rectangle stabbing problem.

References

[GIK02] Daya Ram Gaur, Toshihide Ibaraki, and Ramesh Krishnamurti. Constant Ratio Approximation Algorithms for the Rectangle Stabbing Problem and the Rectilinear Partitioning Problem. *Journal of Algorithms*, 43(1):138–152, April 2002.