Minimizing the Sum of Piecewise Linear Convex Functions

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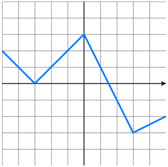
Plan

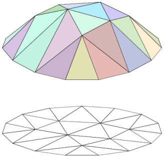
1. Problems & Definitions
2. Properties
3. LP in Low Dimensions
4. Possible Improvements

$\min \sum f_i(a_i \cdot x - b_i)$

Problem. *Given n piecewise linear convex functions $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ of total m breakpoints, and n linear functions $a_i \cdot x - b_i : \mathbb{R}^d \to \mathbb{R}$, find $\min_x \sum_i f_i(a_i \cdot x - b_i)$.*



(a) A 1D pwl function with 4 line segments and 3 breakpoints



(b) A 2D pwl concave function

$f_i(a_i \cdot x - b_i) : \mathbb{R}^d \to \mathbb{R}$ is also piecewise linear convex.

General piecewise linear convex function in $\mathbb{R}^d$

Definition [piecewise linear convex function in $\mathbb{R}^d$].

$$g(x) = \max\{a_1^T x + b_1, \ldots, a_L^T x + b_L\}$$

Every piecewise linear convex function in $\mathbb{R}^d$ can be expressed in this form.[1]
However, observe that in our problem the piecewise linear convex function is not that general. It is a composition of a linear mapping and an 1D piecewise linear convex function.

[1] S.P. Boyd, L. Vandenberghe, **Convex optimization**, Cambridge University Press, Cambridge, UK ; New York, 2004.

$f \circ l \neq g$

Proof. Consider a piecewise linear convex function $g : \mathbb{R}^2 \to \mathbb{R}$. g can be viewed as the maximum of a set of planes in $\mathbb{R}^3$.
Consider a series of points $P = \{p_1, p_2, ..., p_k\}$ on the 2D plane. After applying the linear mapping to P, we will get a sequence of numbers(points in 1D) $P' = \{p'_1, p'_2, ..., p'_k\}$. We assume that P' is non-decreasing. Note that the value of g on P' is always unimodal since g is convex. However, the value of f on P may not be unimodal. Thus the composition of a linear mapping and a pwl convex function in 1D is not equivalent to pwl convex functions in high dimensions. □

A linear time algorithm I

Problem. *Given n piecewise linear convex functions $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R}$ of total m breakpoints, and n linear functions $a_i \cdot x - b_i : \mathbb{R}^d \to \mathbb{R}$, find $\min_x \sum_i f_i(a_i \cdot x - b_i)$.* This can be solve in $O(2^{2^d}(m+n))$ through Megiddo's Low dimension LP algorithm.[2]

Let n_i be the number of line segments in f_i. Note that $\sum_i n_i = m + n$.

We can formulate the optimization problem as the following linear program,

A linear time algorithm II

$$
\begin{aligned}
&\min \sum_{i=1}^{n} f_i \\
s.t. \quad & f_i \geq \alpha_j (a_i \cdot x - b_i) - \beta_j \quad \forall i \in [n], \forall j
\end{aligned}
$$

where $\alpha_j x - \beta_j$ is the j'th line segment on f_i.
There will be $m + n$ constraints in total.

[2] Nimrod Megiddo. Linear programming in linear time when the dimension is fixed. J. ACM, 31(1):114–127, jan 1984.

Megiddo's algorithm I

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https://people.inf.ethz.ch/gaertner/subdir/texts/own_work/chap50-fin.pdf
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The dimension d (in our problem, the dimension of x) is small while the number of constraints are huge. We need only d linearly independent tight constraints to identify the optimal solution x^*. Thus most of the constraints are useless.

For one constraint, how can we know where does x^* locate with respect to it?

Through inquiries. Let $a \cdot x \leq b$ be the constraint. Define 3 hyperplanes, $a \cdot x = c$ where $c \in \{b, b - \varepsilon, b + \varepsilon\}$. Now solve three $d - 1$ dimension linear programming. The largest of the three objective functions tells us where x^* lies with respect to the hyperplane.

Megiddo's algorithm II

Finding the optimal solution x^* is therefore equivalent to the following problem,

Problem [Multidimensional Search Problem]. *Suppose that there exists a point x^* which is not known to us, but there is a oracle that can tell the position of x^* relative to any hyperplane in $\mathbb{R}^d$. Given n hyperplanes, we want to know the position of x^* relative to each of them.*

What about 1 dimension search? A fastest way will be using the linear time median algorithm. We can find the median of n numbers and call the oracle to compare the median with x^*. Thus with $O(n)$ time median finding and one oracle call, we find the relative position of $n/2$ elements relative to x^*.

Megiddo's algorithm III

If we can do similar things in $\mathbb{R}^d$, i.e., there is a method which makes $A(d)$ oracle calls and determines at least $B(d)$ fraction of relative positions, then we can apply this method $\log_{\frac{1}{1-B(d)}} n$ times to find all relative positions.
Note that in 1 dimension, $A(1) = 1$ and $B(1) = 1/2$ (call oracle to compare x^* and the median). In $\mathbb{R}^d$, our oracle is the recursive inquiry.
A trivial method will be iterating on all hyperplanes and calling the oracle on each one, since there is no *median* of a set of hyperplanes in $\mathbb{R}^d$. The complexity recurrence is

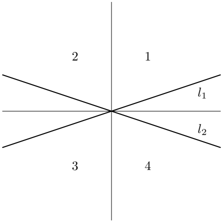
$$T(n, d) = n(3T(n-1, d-1) + O(nd))$$

Note that in this setting $A(d) = 1$ and $B(d) = 1/n$.

Megiddo's algorithm IV

Megiddo designed a clever method where $A(d) = 2^{d-1}$ and $B(d) = 2^{-(2^d-1)}$.

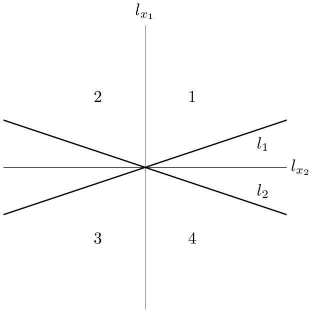
Lemma.



Given two lines through the origin with slopes of opposite sign, knowing which quadrant x^* lies in allows us to locate it with respect to at least one of the lines.

Megiddo's algorithm V

Let l_H be the intersection of hyperplane H and x_1x_2 plane. Compute a partition $S_1 \sqcup S_2 = \mathcal{H}$. $H \in S_1$ iff l_H has positive slope. Otherwise $l_H \in S_2$. We further assume that $|S_1| = |S_2| = n/2$.



Now we have $n/2$ pairs (H_1, H_2), where $H_i \in S_i$. Let l_i be the intersection of H_i and x_1x_2 plane. Let H_{x_i} be the linear combination of H_1 and H_2 s.t. x_i is eliminated.

Megiddo's algorithm VI

By the previous lemma, calling oracle on l_{x_1} and l_{x_2} locate x^* with respect to at least one of H_1 and H_2.

Megiddo's algorithm VII

Input: S_1, S_2 and the pairs.

1. recursively locate x^* respect to $B(d-1)n/2$ hyperplanes(H_{x_i}) with $A(d-1)$ oracle calls in S_1.
2. locate with respect to a $B(d-1)$-fraction of corresponding paired hyperplanes in S_2.
3. There are still $(1 - B(d-1)^2)/2$-fraction of hyperplanes for which we do not know the relative position with x^*. Run this algorithm on these hyperplanes.

This gives the recurrence

$$T(n,d) \leq 3 \cdot 2^{d-1} T(n,d-1) + T((1-2^{1-2^d})n,d) + O(nd)$$

with solution $T(n,d) = O(2^{2^d} n)$.

Zemel's conversion

$$\min \sum_{i=1}^{n} f_i$$
$$s.t. \quad f_i \geq \alpha_j(a_i \cdot x - b_i) - \beta_j \quad \forall i \in [n], \forall j$$

Our linear program has *dimension* $n + d$. **Zemel** showed that this kind of problem can be solved in linear time.

> This is a *d-dimensional search problem* with $n + d$ hyperplanes.

Other algorithms for fixed dimension LP

simplex method	det.	$O(n/d)^{d/2+O(1)}$
Megiddo [24]	det.	$2^{O(2^d)}n$
Clarkson [9]/Dyer [14]	det.	$3^{d^2}n$
Dyer and Frieze [15]	rand.	$O(d)^{3d}(\log d)^d n$
Clarkson [10]	rand.	$d^2 n + O(d)^{d/2+O(1)}\log n + d^4\sqrt{n}\log n$
Seidel [26]	rand.	$d!n$
Kalai [19]/Matoušek, Sharir, and Welzl [23]	rand.	$\min\{d^2 2^d n,\ e^{2\sqrt{d\ln(n/\sqrt{d})}+O(\sqrt{d}+\log n)}\}$
combination of [10] and [19, 23]	rand.	$d^2 n + 2^{O(\sqrt{d\log d})}$
Hansen and Zwick [18]	rand.	$2^{O(\sqrt{d\log((n-d)/d)})}n$
Agarwal, Sharir, and Toledo [4]	det.	$O(d)^{10d}(\log d)^{2d}n$
Chazelle and Matoušek [8]	det.	$O(d)^{7d}(\log d)^d n$
Brönnimann, Chazelle, and Matoušek [5]	det.	$O(d)^{5d}(\log d)^d n$
~~this paper~~ Chan	det.	$O(d)^{d/2}(\log d)^{3d}n$

Figure: Algorithms for LP in low dimensions [3]

Can we use faster fixed dimension LP algorithms to get better complexity?

[3] table stolen from https://dl.acm.org/doi/10.1145/3155312

LP-type problem I

Algorithms for low dim LP are actually solving a more abstract problem.

Definition [LP-type problem]. Given a set S and a function $f : S \to \mathbb{R}$. f satisfies two properties:

- Monotonicity: $\forall A \subseteq B \subseteq S, f(A) \leq f(B) \leq f(S)$.
- Locality: $\forall A \subseteq B \subseteq S$ and $\forall x \in S$, if $f(A) = f(B) = f(A \cup \{x\})$, then $f(A) = f(B \cup \{x\})$.

Linear programs(minimization) are LP-type problems.
$B \subseteq S$ is a basis if $\forall B' \subsetneq B, f(B') < f(B)$. A set of 'useful' constraints in a linear program is a basis.
The combinatorial dimension is the size of the largest basis.
If a LP problem has low dimension, then its combinatorial dimension is low. **What about the converse?**

LP-type problem II

$$\begin{aligned} &\min \sum_{i=1}^{n} f_i \\ s.t. \quad & f_i \geq \alpha_j (a_i \cdot x - b_i) - \beta_j \quad \forall i \in [n], \forall j \\ & \dots \end{aligned}$$

Does our LP has low combinatorial dimension?
No. A basis contains at least n constraints since otherwise some f_i is unbounded.

Problem. *Is it possible to formulate the pwl convex minimization problem as an LP-type problem with low combinatorial dimension?*

Aggregate the pwl convex functions

The sum of pwl convex functions are still pwl convex.
If we can compute $F = \sum f_i$ in $O(m)$ and the number of line segments on F is also $O(m)$, then the corresponding LP will have low combinatorial dimension.

$$\begin{aligned} \min \quad & F \\ s.t. \quad & F \geq \alpha_j \cdot x - \beta_j \quad \forall j \\ & \cdots \end{aligned}$$

However, this is not possible for general pwl convex functions in $\mathbb{R}^d$.[4]

[4] see this blog post for detail.

pseudocode

```
sort vertices in G in such that deg(v_1) ≥ ··· ≥ deg(v_n)
for i ∈ [n]:
    for each vertex u ∈ N(v_i):
        let U[v] = ∅ for all v.
        for each vertex w ∈ N(u) that is not v_i:
            add u to U[w].
    for all vertex w ∈ V that is not v_i:
        if |U[w]| ≥ ℓ:
            output tuple (v_i, w, U[w])
    G = G − v_i
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Figure: An $O(m\alpha(G))$ algorithm for finding all colored $K_{2,\ell'}$ for $\ell' \geq \ell$